Jumps in speeds of hereditary properties

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Model Theory and Counting

Given a theory T and a cardinal κ, let $I(T, \kappa)$ be the number of non-isomorphic models of T of cardinality κ.

Question

What are the possible behaviors of the function $I(T, \kappa)$?

• Model theoretic dividing lines were developed by Shelah to answer this question when $\kappa > \aleph_0$. Subsequent cases resolved by Hart, Hrushovski, and Laskowski.

• A main takeaway: the behavior of $I(T, \kappa)$ is related to structural properties of models of T.

Question

What are finite analogues of this question?

Hereditary Properties

Suppose $\mathcal{L}$ is a finite relational language. Throughout, theories are possibly incomplete.

Definition

A *hereditary $\mathcal{L}$-property* is a class of $\mathcal{L}$-structures closed under substructures and isomorphism.

Fact

Suppose $\mathcal{H}$ is a class of $\mathcal{L}$-structures. $\mathcal{H}$ is a hereditary $\mathcal{L}$-property if and only if $\mathcal{H}$ is the class of models of a universal $\mathcal{L}$-theory $T_{\mathcal{H}}$.

$\{$ hereditary properties $\} \leftrightarrow \{$ universal theories $\}$

Speeds

Notation: Given $n \geq 1$, $[n] = \{1, \ldots, n\}$.

For a hereditary $\mathcal{L}$-property $\mathcal{H}$, define

$$\mathcal{H}_n = \{\mathcal{M} \in \mathcal{H} : \mathcal{M} \text{ has domain } [n]\}$$

Definition

The *speed* of $\mathcal{H}$ is the function $n \mapsto |\mathcal{H}_n|$. Define $I(n, \mathcal{H}) := |\mathcal{H}_n|$.

(note $I(n, \mathcal{H})$ counts elements in $\mathcal{H}_n$ *not up to isomorphism*).

Equivalently, $I(n, \mathcal{H})$ is the number of distinct quantifier-free types on n-variables appearing in any model of $T_{\mathcal{H}}$.

Question

Question

What are the possible behaviors of $I(n, \mathcal{H})$? Is $I(n, \mathcal{H})$ a good "invariant" of the universal theory $T_{\mathcal{H}}$?

This question is about counting finite models of a universal theory.

This question is about counting the number of quantifier-free types in models of a universal theory.

If $\mathcal{L} = \{E(x, y)\}$, a hereditary $\mathcal{L}$-property consisting entirely of graphs is a *hereditary graph property*.

Cool Theorem

Question

What are the possible behaviors of the function $I(n, \mathcal{H})$ when $\mathcal{H}$ is a hereditary graph property?

It turns out there are discrete possibilities:

Theorem 1 (Balogh-Bollobás-Weinrich, Bollobás-Thomason, Alon-Balogh-Bollobás-Morris)

If $\mathcal{H}$ is a hereditary graph property, then one of the following holds.

1. $|\mathcal{H}_n| = \sum_{i=1}^{k} p_i(n) i^n$ for rational polynomials $p_1, \ldots, p_k$ (poly/exp).
2. $|\mathcal{H}_n| = n^{(1-\frac{1}{k}+o(1))n}$ for some integer $k > 1$. (factorial).
3. $n^{n(1-o(1))} \leq |\mathcal{H}_n| \leq 2^{n^{2-\epsilon}}$ for some $C, \epsilon > 0$. (penultimate).
4. $|\mathcal{H}_n| = 2^{Cn^2(1+o(1))}$ for some $C > 0$. (exponential in n^2). (IP).

Problem

Fact: The proof of Theorem 1 connects the behavior of $I(n, \mathcal{H})$ to structural properties of models of $T_{\mathcal{H}}$.

Problem

Generalize Theorem 1 to the setting of arbitrary finite relational languages.

For the rest of the talk, $\mathcal{L}$ is a finite relational language of maximum arity r.

Previous Work

Theorem (T.; independently Falgas-Ravry, O'Connell, and Strömberg, 2016)

Suppose $r \geq 1$ and $\mathcal{H}$ is a hereditary $\mathcal{L}$-property, then either

(i) $|\mathcal{H}_n| = 2^{o(n^r)}$.

(ii) $|\mathcal{H}_n| = 2^{Cn^r(1+o(1))}$, *some* $C > 0$.

Generalizes hypergraph case by Alekseev, Bollobás-Thomason.

Theorem (T., 2017)

Suppose $r \geq 1$ and $\mathcal{H}$ is a hereditary $\mathcal{L}$-property, then either

(i) $|\mathcal{H}_n| \leq 2^{n^{r-\epsilon}}$, *some* $C, \epsilon > 0$. *(finite VC_r-dimension).*

(ii) $|\mathcal{H}_n| = 2^{Cn^r(1+o(1))}$, *some* C. *(infinite VC_r-dim).*

Generalizes $r = 2$ case by Alon-Balogh-Bollobás-Morris.

Previous Work: Sum up

Theorem (T., 2016)

Suppose $r \geq 1$ and $\mathcal{H}$ is a hereditary $\mathcal{L}$-property, then either

(i) $|\mathcal{H}_n| \leq n^k$. *(poly).*

(ii) $2^{Cn} \leq |\mathcal{H}_n| \leq 2^{n^{r-\epsilon}}$, *some C, ϵ. (???).*

(iii) $|\mathcal{H}_n| = 2^{Cn^r(1+o(1))}$, *some C. (exponential in n^r). (infinite VC_r-dim).*

For comparison:

Theorem 1 (Graphs Case, abbreviated)

If $\mathcal{H}$ is a hereditary graph property, then either

1. $|\mathcal{H}_n| = \sum_{i=1}^{k} p_i(n) i^n$ for rational polynomials $p_1, \ldots, p_k$ (poly/exp).
2. $|\mathcal{H}_n| = n^{(1-\frac{1}{k}+o(1))n}$ for some integer $k > 1$. (factorial).
3. $n^{n(1-o(1))} \leq |\mathcal{H}_n| \leq 2^{n^{2-\epsilon}}$ for some $C, \epsilon > 0$. (penultimate).
4. $|\mathcal{H}_n| = 2^{Cn^2(1+o(1))}$ for some $C > 0$. (exponential in n^2).(IP).

Main Theorem

Theorem (Laskowski-T., 2018)

For any hereditary $\mathcal{L}$-property $\mathcal{H}$ one of the following holds.

1. $|\mathcal{H}_n| = \sum_{i=1}^{k} p_i(n) i^n$ *for rational polynomials* $p_1, \dots, p_k$. *(poly/exp).*
2. $|\mathcal{H}_n| = n^{n(1-\frac{1}{k}-o(1))}$ *for some* $k > 1$. *(factorial).*
3. $|\mathcal{H}_n| \geq n^{(1-o(1))n}$.

Melding these:

Corollary (Laskowski-T., T.)

For any hereditary $\mathcal{L}$-property $\mathcal{H}$ one of the following holds.

1. $|\mathcal{H}_n| = \sum_{i=1}^{k} p_i(n) i^n$ for rational polynomials $p_1, \dots, p_k$. (poly/exp).
2. $|\mathcal{H}_n| = n^{(1-\frac{1}{k}+o(1))n}$ for some integer $k > 1$. (factorial).
3. $n^{n(1-o(1))} \leq |\mathcal{H}_n| \leq 2^{n^{r-\epsilon}}$ for some $C, \epsilon > 0$. (penultimate).
4. $|\mathcal{H}_n| = 2^{Cn^r(1+o(1))}$. (exponential in n^r). (infinite VC_r-dim).

Type spaces and arrays

Suppose $\mathcal{M}$ is an $\mathcal{L}$-structure and $A \subseteq M$.

Definition

Let $S_{\bar{x}}(A)$ denote the set of complete quantifier-free types over A in the variables $\bar{x}$.

Note that if A is finite then so is $S_{\bar{x}}(A)$.

Definition

Given an integer $m \geq 1$, an *m-array in $\mathcal{M}$* is a set $\{\bar{d}_i : i \in [m]\}$ of pairwise disjoint tuples from M^k for some $k \geq 1$.

Definition

We say $p(\bar{x}) \in S_{\bar{x}}(A)$ *supports an m-array* if the set of realizations of $p(\bar{x})$ in $\mathcal{M}$ contains an m-array.

Unbounded Arrays

Recall "theory"= possibly incomplete theory.

Definition

A theory T has *unbounded arrays* if there is a finite tuple of variables $\bar{x}$ such that for arbitrarily large m, N:
There is $\mathcal{M} \models T$ and a finite $A \subseteq M$ such that

$$|\{p(\bar{x}) \in S_{\bar{x}}(A) : p(\bar{x}) \text{ supports an } m\text{-array}\}| \geq N.$$

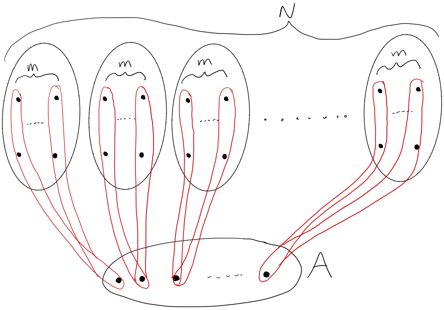
We say $\mathcal{H}$ has *unbounded arrarys* if $T_{\mathcal{H}}$ does.

Unbounded Arrays: Example

Let $\mathcal{L} = \{R(x, y, z)\}$. A 3-*uniform hypergraph* is an $\mathcal{L}$-structure where R is symmetric and irreflexive.

Let $\mathcal{H} = \{$linear 3-uniform hypergraphs$\}$ (every two vertices are in at most one edge).

$\mathcal{H}$ has unbounded arrays since for any m, N this is in $\mathcal{H}$:



Unbounded arrays imply "fast growth"

Proposition

If $\mathcal{H}$ has unbounded arrays then $|\mathcal{H}_n| \geq n^{n(1-o(1))}$.

Idea: if $\mathcal{H}$ has unbounded arrays, then for all $m \ll n$, we can code equipartitions of $[n]$ into pieces of size m (more or less).

What if we do not have unbounded arrays?

We will split into two cases:

• If $\mathcal{H}$ has unbounded arrays, then $|\mathcal{H}_n| \geq n^{n(1-o(1))}$.

• If $\mathcal{H}$ does not have unbounded arrays, $\mathcal{H}$ has "easy to understand structure."

What does "easy to understand structure" mean?

Mutual algebraicity

Definition

Given an $\mathcal{L}$-formula $\phi(\bar{z})$ and a $\mathcal{L}$-structure $\mathcal{M}$, we say $\phi(\bar{z})$ is *k-mutually algebraic in $\mathcal{M}$* if for every non-trivial partition $\bar{z} = \bar{x} \wedge \bar{y}$,

$$\mathcal{M} \models \forall \bar{x} \exists^{<k} \bar{y} \phi(\bar{x}\bar{y}).$$

Example: Let $\mathcal{L} = \{R(x, y, z)\}$ and let $\mathcal{M}$ be a 3-uniform hypergraph where every vertex is in at most 1 edge. Then $R(x_1, x_2, x_3)$ is 2-mutually algebraic in $\mathcal{M}$. E.g.

$$\mathcal{M} \models \forall x_1 x_2 \exists^{<2} x_3 R(x_1, x_2, x_3),$$

$$\mathcal{M} \models \forall x_1 \exists^{<2} x_2 x_3 R(x_1, x_2, x_3),$$

Mutual algebraicity

Definition

An $\mathcal{L}$-structure $\mathcal{M}$ is *mutually algebraic* if there is an integer k and a finite set Δ of quantifier-free formulas (with parameters) such that

- Every relation $R \in \mathcal{L}$ is equivalent in $\mathcal{M}$ to a boolean combination of formulas from Δ, and
- Every formula in Δ is k-mutually algebraic in $\mathcal{M}$.

Definition

A theory T is *mutually algebraic* if every $\mathcal{M} \models T$ is mutually algebraic.

Mutual algebraicity

If T is mutually algebraic then T is stable. In fact,

Theorem (Laskowski, 2012)

A complete theory T is weakly minimal and trivial if and only if it is mutually algebraic.

Definition

A hereditary $\mathcal{L}$-property $\mathcal{H}$ is *mutually algebraic* if $T_{\mathcal{H}}$ is.

Counting mutually algebraic properties

Theorem (Laskowski-T.)

Suppose $\mathcal{H}$ is mutually algebraic. Then one of the following holds.

1. $|\mathcal{H}_n| = \sum_{i=1}^{k} p_i(n) i^n$ *for some rational polynomials* $p_1, \ldots, p_k$,
2. $|\mathcal{H}_n| = n^{(1-\frac{1}{k}-o(1))n}$ *for some integer* $k > 1$, *or*
3. $|\mathcal{H}_n| \geq n^{(1-o(1))n}$.

Proof Idea: Analyze behavior of "connected components" with respect to mutually algebraic formulas.

New Theorem

Theorem (Laskowski-T., 2018)

A theory T in a finite relational language is mutually algebraic if and only if T does not have unbounded arrays.

Proof uses quantifier-free versions of tools from stability theory and previous results of Laskowski.

Corollary: the factorial range

Corollary (Laskowski-T.)

For any hereditary $\mathcal{L}$-property $\mathcal{H}$ one of the following holds.

1. $|\mathcal{H}_n| = \sum_{i=1}^{k} p_i(n) i^n$ for some rational polynomials $p_1, \ldots, p_k$,
2. $|\mathcal{H}_n| = n^{(1-\frac{1}{k}-o(1))n}$ for some $k > 1$, or
3. $|\mathcal{H}_n| \geq n^{(1-o(1))n}$.

Proof.

If $\mathcal{H}$ has unbounded arrays then $|\mathcal{H}_n| \geq n^{n(1-o(1))}$.
So assume $\mathcal{H}$ does not have unbounded arrays.
Then $\mathcal{H}$ is mutually algebraic, so we apply the counting theorem for mutually algebraic properties. □

Thanks for listening!