Coding and interpreting structures

Julia F. Knight[1]

University of Notre Dame

[1] Joint work with Alexandra Soskova, Stefan Vatev, Andrey Morozov, Russell Miller, Rachael Alvir, Rose Weisshaar, Wesley Calvert, Valentina Harizanov, and Grant Goodman

Coding and decoding

There are familiar ways of coding one structure in another, and for coding members of one class of structures in those of another class.

Sometimes the coding is effective. Assuming this, it is interesting when decoding is also effective. It is also interesting when decoding is difficult.

We consider some formal notions that describe coding and decoding, and we test the notions in some examples.

Conventions

1. Languages are computable.
2. Structures have universe ω.
3. Fixing the language L, we identify an L-structure $\mathcal{A}$ with $D(\mathcal{A})$, and we identify *that* with a function in 2^ω.
4. Classes K are closed under isomorphism.

Borel embeddings

Definition (Friedman-Stanley, 1989). For classes K, K', we say that K is *Borel embeddable* in K, and we write $K \leq_B K'$, if there is a Borel function $\Theta : K \to K'$ s.t. for $\mathcal{A}, \mathcal{B} \in K$, $\mathcal{A} \cong \mathcal{B}$ iff $\Theta(\mathcal{A}) \cong \Theta(\mathcal{B})$.

Note: A Borel embedding $\Theta : K \to K'$ represents a uniform explicit method for coding structures from K in structures from K'.

On top

Theorem. The following classes lie on top under $\leq_B$.

1. undirected graphs (Lavrov, 1963; Nies, 1996; Marker, 2002)
2. fields (Friedman-Stanley, 1989;
   R. Miller-Poonen-Schoutens-Shlapentokh, 2018)
3. 2-step nilpotent groups (Mekler, 1981; Mal'tsev, 1949)
4. linear orderings (Friedman-Stanley, 1989)

Turing computable embeddings

Definition (Calvert-Cummins-K-S. Miller (Quinn), 2004). For classes K, K', we say that K is *Turing computably embedded* in K', and we write $K \leq_{tc} K'$, if there is a Turing operator $\Theta : K \to K'$ s.t. for all $\mathcal{A}, \mathcal{B} \in K$, $\mathcal{A} \cong \mathcal{B}$ iff $\Theta(\mathcal{A}) \cong \Theta(\mathcal{B})$.

Note: The notion of Turing computable embedding captures in a precise way the idea of uniform effective coding.

On top

Theorem. The following classes lie on top under $\leq_{tc}$.

1. undirected graphs
2. fields
3. 2-step nilpotent groups
4. linear orderings

Proof: The Borel embeddings of Friedman-Stanley, Miller-Poonen-Schoutens-Shlapentokh, Lavrov, Nies, Marker, Mekler, and Mal'tsev are, in fact, Turing computable.

Marker's embedding

For a directed graph $\mathcal{A}$, the undirected graph $M(\mathcal{A})$ consists of:

1. a point b_a with a triangle attached, for each $a \in \mathcal{A}$,
2. a point $p_{(a,a')}$ for each ordered pair (a, a') from $\mathcal{A}$, where $p_{(a,a')}$ is connected to b_a directly, and to $b_{a'}$ with one stop,
3. further elements for each pair (a, a') that, together with $p_{(a,a')}$, form

$$\begin{cases} \text{a square} & \text{if } a \to a' \\ \text{a pentagon} & \text{otherwise} \end{cases}$$

Remark. For structures $\mathcal{A}$ with more relations, the same idea works.

Effective decoding

Definition. We say that $\mathcal{A}$ is *Medvedev reducible* to $\mathcal{B}$, and we write $\mathcal{A} \leq_s \mathcal{B}$ if there is a Turing operator that takes copies of $\mathcal{B}$ to copies of $\mathcal{A}$.

If $\mathcal{A}$ is coded in $\mathcal{B}$, then a Medvedev reduction of $\mathcal{A}$ to $\mathcal{B}$ represents an effective decoding procedure.

For classes K and K', suppose that $K \leq_{tc} K'$ via Θ. A uniform effective decoding procedure is a Turing operator Φ s.t. for all $\mathcal{A} \in K$, $\mathcal{A} \leq_s \Theta(\mathcal{A})$ via Φ.

Decoding via nice defining formulas

Fact: For Marker's embedding M, we have finitary existential formulas that, for all directed graphs $\mathcal{A}$, define the following.

1. the set D of b_a connected to a triangle,
2. the set of ordered pairs $(b_a, b_{a'})$ s.t. the special point $p_{(a,a')}$ is part of a square,
3. the set of ordered pairs $(b_a, b_{a'})$ s.t. the special point $p_{(a,a')}$ is part of a pentagon,

This guarantees a uniform effective procedure that, given any copy of $\Theta(\mathcal{A})$, computes a copy of $\mathcal{A}$. We have uniform effective decoding.

Effective interpretation

Harrison-Trainor, Melnikov, R. Miller, and Montalbán defined a very general kind of interpretation that guarantees effective decoding.

Definition. A structure $\mathcal{A} = (A, R_i)$ is *effectively interpreted* in a structure $\mathcal{B}$ if there exist a set $D \subseteq \mathcal{B}^{<\omega}$, and relations $\sim$ and R_i^* on D, s.t.

1. $(D, R_i^*)/_\sim \cong \mathcal{A}$,
2. D, $\pm \sim$, and $\pm R_i^*$ are all defined by computable Σ_1 formulas with no parameters. (In case the language of $\mathcal{A}$ is infinite, the definitions of $\pm R_i^*$ are effectively determined.)

Computable functor

Definition. A *computable functor* from $\mathcal{B}$ to $\mathcal{A}$ is a pair of Turing operators Φ, Ψ s.t.

1. Φ takes copies of $\mathcal{B}$ to copies of $\mathcal{A}$,
2. Ψ takes isomorphisms between copies of $\mathcal{B}$ to isomorphisms between the corresponding copies of $\mathcal{A}$, so as to preserve identity and composition.

More precisely, Ψ is defined on triples $(\mathcal{B}_1, f, \mathcal{B}_2)$, where $\mathcal{B}_1, \mathcal{B}_2$ are copies of $\mathcal{B}$ with $\mathcal{B}_1 \cong_f \mathcal{B}_2$.

Equivalence

Theorem (H-TMMM, 2017). $\mathcal{A}$ is effectively interpreted in $\mathcal{B}$ iff there is a computable functor from $\mathcal{B}$ to $\mathcal{A}$.

Note: In the proof, it is important that D consist of tuples of arbitrary arity.

Disturbing example

Proposition. If $\mathcal{A}$ is computable, then it is effectively interpreted in all structures $\mathcal{B}$.

Proof: Let $D = \mathcal{B}^{<\omega}$. Let $\bar{b} \sim \bar{c}$ if $\bar{b}, \bar{c}$ are tuples of the same length. For simplicity, suppose $\mathcal{A} = (\omega, R)$, where R is binary. If $\mathcal{A} \models R(m, n)$, then $R^*(\bar{b}, \bar{c})$ for all $\bar{b}$ of length m and $\bar{c}$ of length n.

Natural questions, answered by Kalimullin

Questions.

1. If $\mathcal{A} \leq_s \mathcal{B}$, must $\mathcal{A}$ be effectively interpreted in $\mathcal{B}$?
2. If $\mathcal{A}$ is effectively interpreted in $(\mathcal{B}, \bar{b})$, is it effectively interpreted in $\mathcal{B}$?

Proposition (Kalimullin, 2010). The answer to both questions is "No."

Mal'tsev embedding of fields in groups

For a field F, let $H(F)$ be the *Heisenberg group*, consisting of the matrices over F of form

$$\begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$$

Theorem (Mal'tsev). There is a copy of F defined in $H(F)$ with parameters.

Definition of F in $H(F)$

Let u, v be a non-commuting pair in $H(F)$. Then $(D, +, \cdot_{(u,v)})$ is a copy of F, where

1. D is the group center—$x \in D$ iff $[x, u] = 1$ and $[x, v] = 1$,
2. $x + y = z$ if $x * y = z$, where $*$ is the group operation,
3. $x \cdot_{(u,v)} y = z$ if there exist x', y' s.t.

   $[x', u] = [y', v] = 1$, $[x', v] = x$, $[u, y'] = y$, and $[x', y'] = z$.

Uniform effective decoding

Definability: There are finitary existential formulas, with no parameters, that define D and the relation $+$ and its complement, and there are finitary existential formulas, with an arbitrary non-commuting pair (u, v) as parameters, that define the relation $\cdot$ and its complement.

Corollary (Morozov). There is a uniform Turing operator Φ such that for all fields F, $F \leq_s H(F)$ via Φ.

Proof We search for a non-commuting pair (u, v) in $H(F)$, and then use Mal'tsev's definitions to compute a copy of F.

Half of computable functor

The Turing operator Φ is one half of a computable functor from $H(F)$ to F.

Question. Is F effectively interpreted in $H(F)$—by formulas without parameters?

The answer is "Yes."

Natural isomorphism from F to copy defined in $H(F)$

We write $h(a, b, c)$ for the matrix

$$\begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$$

For a non-commuting pair (u, v), where $u = h(u_1, u_2, u_3)$ and $v = h(v_1, v_2, v_3)$, let

$$\Delta_{(u,v)} = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

Morozov. The function $x \to h(0, 0, x \cdot_F \Delta_{(u,v)})$ is an isomorphism from F to $F_{(u,v)}$.

Definable isomorphisms

Lemma (Morozov). Let (u, v) and (u', v') be non-commuting pairs in $G \cong H(F)$. Let $F_{(u,v)}$ and $F_{(u',v')}$ be the copies of F defined in G with these pairs of parameters. There is an isomorphism $g_{(u,v),(u'v')}$ from $F_{(u,v)}$ onto $F_{(u',v')}$, defined in G by an existential formula with parameters u, v, u', v'.

Proof. Note that $h(0, 0, \Delta_{(u,v)}) = 1_{(u,v)}$, the multiplicative identity in $F_{(u,v)}$. Let $g_{(u,v),(u',v')}(x) = y$ iff $x = 1_{(u,v)} \cdot_{(u',v')} y$. $\varphi(x)$ saying that

Computable functor

Morozov. There is a computable functor Φ, Ψ from $H(F)$ to F.

Φ: For $G \cong H(F)$, $\Phi(G))$ is the copy of F obtained by taking the first non-commuting pair (u, v) in G and forming $(D, +, \cdot_{(u,v)})$.

Ψ: Take (G_1, f, G_2), where $G_i \cong H(F)$, and $G_1 \cong_f G_2$. Let (u, v), (u', v') be the first non-commuting pairs in G_1, G_2, respectively. Let $F_{((u',v')}$ and $F_{(f(u),f(v))}$ be the copies of F defined in G_2 using the parameters (u', v') and $(f(u), f(v))$, respectively. Let h be the isomorphism from $F_{(f(u),f(v))}$ onto $F_{(u',v')}$ defined in G_2 with parameters $f(u), f(v), u', v'$. Let f' be the restriction of f to the center of G_1. Then $\Psi(G_1, f, G_2) = h \circ f'$.

Corollary (Alvir-Calvert-Harizanov-K-Miller-Morozov-Soskova-Weisshaar). F is effectively interpreted in $H(F)$.

Finding an interpretation directly

Proposition (Goodman-K-Miller). The following is an effective interpretation.

1. D is the set of (u, v, x) s.t. $[u, v] \neq 1$ and $[x, u] = [x, v] = 1$.
2. $(u, v, x) \sim (u', v', x')$ holds if Morozov's isomorphism from $F_{(u,v)}$ to $F_{(u',v')}$ takes x to x'.
3. $+^*((u, v, x), (u', v', y), (u'', v'', z))$ holds if there exist y', z' s.t. $(u, v, y') \sim (u', v', y)$, $(u, v, z') \sim (u'', v'', z)$, and $x * y' = z'$.
4. $\cdot^*((u, v, x), (u', v', y), (u'', v'', z))$ holds if there exist y', z' s.t. $(u, v, y') \sim (u', v', y)$, $(u, v, z') \sim (u'', v'', z)$, and $x \cdot_{(u,v)} y' = z'$.

Generalizing

Proposition. Suppose $\mathcal{A}$ is defined in $\mathcal{B}$ by computable Σ_1 formulas with parameters $\bar{b}$. For $\bar{c}$ in the orbit of $\bar{b}$, let $\mathcal{A}_{\bar{c}}$ be the copy of $\mathcal{A}$ defined with parameters $\bar{c}$ replacing $\bar{b}$. The following conditions suffice for an effective interpretation of $\mathcal{A}$ in $\mathcal{B}$:

1. The orbit of $\bar{b}$ in $\mathcal{B}$ is defined by a computable Σ_1 formula $\varphi(\bar{x})$.
2. There is a computable Σ_1 formula $\psi(\bar{u}, \bar{v}, x, y)$ s.t. for any $\bar{c}, \bar{d}$ in the orbit of $\bar{b}$, defines an isomorphism $f_{\bar{c},\bar{d}}$ from $\mathcal{A}_{\bar{c}}$ onto $\mathcal{A}_{\bar{d}}$.
3. The family of isomorphisms $f_{\bar{c},\bar{d}}$ preserves identity and composition.

Trivial example

Let A be a non-c.e. set that contains 0. Let S be the family of all finite sets that do not contain 0. Let $\mathcal{A} = G_A^\infty$, let $\mathcal{B} = G_{S \cup \{A\}}^\infty$, and let $\mathcal{C} = G_S^\infty$. The daisies in these graphs are directed, so that all elements of a given daisy are existential defined in terms of the center. Let b be an element of $\mathcal{B}$ that is the center of a copy of G_A.

1. There is a copy of $\mathcal{A}$ defined in $(\mathcal{B}, b)$ by computable Σ_1-formulas—we use tuples of different arity to produce different copies of S_A.
2. There are further computable Σ_1 formulas $\varphi(u)$ and $\psi(u, v, x, y)$ satisfying the conditions of the previous result.
3. Hence, $\mathcal{A}$ is effectively interpreted in $\mathcal{B}$.

A possible non-example

Consider countable algebraically closed fields C of characteristic 0, $SL_2(C)$. We can define C in $SL_2(C)$ using existential formulas with parameters $p = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $q = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$.

There are old model theoretic results that give definability without parameters—by elementary first order formulas of some complexity. This isn't good enough.

Interpretations by more complicated formulas

Harrison-Trainor, Miller, and Montalbán defined Borel versions of the notion of effective interpretation and computable functor.

1. For a *Borel interpretation* of $\mathcal{A}$ in $\mathcal{B}$, the set $D \subseteq B^{<\omega}$, and the relations R_i and $\sim$ on D are defined by formulas of $L_{\omega_1 \omega}$.
2. For a *Borel functor* from $\mathcal{B}$ to $\mathcal{A}$, the operators Φ, Ψ are Borel.

Theorem (H-TMM). There is a Borel interpretation of $\mathcal{A}$ in $\mathcal{B}$ iff there is a Borel functor from $\mathcal{B}$ to $\mathcal{A}$.

Embedding of (directed) graphs in orderings

Friedman and Stanley determined a Turing computable embedding $L : G \to L(G)$, where $L(G)$ is a sub-ordering of $\mathbb{Q}^{<\omega}$ under the lexicographic ordering. To specify the elements of $L(G)$, we need some preliminaries.

1. Let $(A_n)_{n\in\omega}$ be an effective partition of $\mathbb{Q}$ into disjoint dense sets.
2. Let $(t_n)_{n\in\omega}$ be a computable list of the atomic types in the language of graphs.

Definition of L(G). For a (directed) graph G, $L(G)$ is the set of tuples $r_0 q_1 r_1 \ldots r_{n-1} q_n r_n k \in \mathbb{Q}^{<\omega}$ s.t. for $i < n$, $r_i \in A_0$, $r_n \in A_1$, and for some $a_1, \ldots, a_n \in G$, satisfying t_m, $q_i \in A_{a_i}$ and $k < m$.

Graphs and linear orderings

Graphs and linear orderings both lie on top under Turing computable embeddings.

Graphs also lie on top under effective interpretation.

Question: What about linear orderings?

Graph not Medvedev reducible to an ordering

Proposition (K-Soskova-Vatev). There is a graph G s.t. $G \not\leq_s L$ for any linear ordering L.

Proposition (K-Soskova-Vatev). There is a graph G s.t. $G \not\leq_s L'$ for any linear ordering L.

This pattern stops

Proposition. For any graph G, any fixed copy G_0, there is a linear ordering L s.t. the $\exists_3$ theory of L computes G_0. It follows that G is effectively interpreted in L'', and it is interpreted in L using computable Σ_3 formulas.

No uniform interpretation of G in $L(G)$

Theorem (K-Soskova-Vatev, Harrison-Trainor-Montalbán). There are not $L_{\omega_1\omega}$-formulas that, for all graphs G, interpret G in $L(G)$.

Outline of proof by K-Soskova-Vatev: We think of an ordering as a directed graph. We show the following.

Proposition.

A. ω_1^{CK} is not interpreted in $L(\omega_1^{CK})$ using computable infinitary formulas.

B. For all X, ω_1^X is not interpreted in $L(\omega_1^X)$ using X-computable infinitary formulas.

Problems

1. For the Friedman-Stanley embedding of graphs in linear orderings, how difficult is it to recover G from $L(G)$?

2. Is there some other Turing computable embedding Θ of graphs in orderings, for which there *are* $L_{\omega_1\omega}$ formulas that, for all graphs G, define an interpretation of G in $\Theta(G)$?