Correspondences and stability

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Some functional analysis

- Fix a complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and consider $B(H)$, the set of all bounded operators on H.
- Recall that a linear map $A : H \to H$ is said to be bounded if the operator norm of A is finite i.e.

$$\|A\| := \sup\{\frac{\|Ax\|}{\|x\|} : x \in H, x \neq 0\} < \infty.$$

  These are the continuous linear operators on H.
- $B(H)$ has a natural complex algebra structure with addition, scalar multiplication, composition as multiplication and one additional operation, the adjoint $*$, defined as follows: for $A \in B(H)$ and $x, y \in H$,

$$\langle Ax, y\rangle = \langle x, A^* y\rangle.$$

Operator algebras

- An operator algebra is a closed *-subalgebra of $B(H)$; the unit will always be included although non-unital algebras are also interesting.
- There are two topologies we are going to care about.
- The first is the topology induced by the operator norm; closed *-subalgebras are C*-algebras.
- The second is the weak operator topology; the closed *-subalgebras are von Neumann algebras (vNa).
- For this talk we will only care about particular types of vNa's so let me define those.

Tracial von Neumann algebras

If M is an operator algebra then we say $\tau : M \to C$ is a trace on M if τ is a complex linear map on M satisfying:

1. (positive) $\tau(a^*a) \geq 0$ for all $a \in M$,
2. (trace condition) $\tau(ab) = \tau(ba)$ for all $a, b \in M$,
3. (normal) $\tau(\sup \mathcal{P}) = \sup_{p \in \mathcal{P}} \tau(p)$ for all sets of projections $\mathcal{P}$ and we will also assume
4. (faithful) for all $a \in M$, $\tau(a^*a) = 0$ iff $a = 0$, and
5. (normalized) $\tau(1) = 1$.

If τ is a trace on M, we call the pair (M, τ) a tracial algebra.

Tracial von Neumann algebras, cont'd

- $M_n(C) = B(C^n)$ has a trace; $\tau(A) = Tr(A)/n$.
- $L^\infty([0,1])$, bounded Lebesgue integrable complex-valued functions on $[0,1]$;

$$\tau(f) = \int_0^1 f\,dx.$$

- $B(H)$ for an infinite-dimensional H is a vNa but is not tracial.

Fact

If (M, τ) is a tracial algebra then $\|a\|_2 = \sqrt{\tau(a^*a)}$ defines a norm on M. If the operator norm unit ball of M is complete wrt $\|\cdot\|_2$ then M is a vNa.

The standard representation

- Suppose (M, τ) is a tracial algebra.
- Let $L^2(M, \tau)$ be the Hilbert space obtained by completing M wrt $\| \cdot \|_2$. This is called the standard representation of M.
- Notice that M acts naturally on $L^2(M, \tau)$ on the left and right by multiplication. So $L^2(M, \tau)$ has a natural (Hilbert) bimodule structure. If we want to emphasize M we say it is an M-bimodule or correspondence.

Property T

- Here is the notion that got us interested in looking at the theory of correspondences; it is the vNa analogue of Kazhdan's property T for unitary group representations.
- We say that a II_1 factor (read tracial vNa if you don't know what this is) M has property T if for every $\epsilon > 0$ there is a finite $F \subseteq M$ and $\delta > 0$ such that if H is an M-bimodule, $\xi \in H$ is a unit vector and $\|[x, \xi]\| \leq \delta$ for all $x \in F$ then there is a central vector $\eta \in H$ such that $\|\eta - \xi\| \leq \epsilon$.
- This seems to say that central vectors form a definable set at least in the right logic.

One slide on continuous model theory

A language $\mathcal{L}$ has sorts, functions and relations. An $\mathcal{L}$-structure is an interpretation of the language subject to the following:

- Sorts are complete, bounded metric spaces with designated metrics.
- Functions are defined on sorts and are uniformly continuous; the language knows the uniform continuity modulus.
- Relations are defined on sorts, bounded and real-valued. They are also uniformly continuous and this is known to the language.
- Terms are formed by composing functions; formulas are defined inductively by:
  - $R(\tau_1, \ldots, \tau_n)$ where R is a relation and $\tau_1, \ldots, \tau_n$ are terms,
  - if $f\colon R^n \to R$ is continuous and $\varphi_1, \ldots, \varphi_n$ are formulas then $f(\varphi_1, \ldots, \varphi_n)$ is a formula, and
  - $\sup_x \varphi$ and $\inf_x \varphi$ are formulas if φ is.

Ultraproducts

- Suppose $\mathcal{U}$ is an ultrafilter on a set I and $\bar{r} = \langle r_i : i \in I \rangle$ is an I-indexed family of real numbers. We define the ultralimit of $\bar{r}$ with respect to $\mathcal{U}$ as follows:

$$\lim_{i \to \mathcal{U}} r_i = r \text{ iff for every } \epsilon > 0, \{i \in I : |r - r_i| < \epsilon\} \in \mathcal{U}.$$

- If (X_i, d_i) is an I-indexed sequence of uniformly bounded metric spaces. Define the pseudo-metric d on $\prod_{i \in I} X_i$ as follows:

$$d(\bar{x}, \bar{y}) = \lim_{i \to \mathcal{U}} d_i(x_i, y_i).$$

- The metric ultraproduct of the X_i's with respect to $\mathcal{U}$, $\prod_{\mathcal{U}} X_i$, is the metric space obtained by quotienting $\prod_{i \in I} X_i$ by d.

Ultraproducts, cont'd

Suppose that $\mathcal{L}$ is a language and M_i is an $\mathcal{L}$-structure for each $i \in I$; fix $\mathcal{U}$ an ultrafilter on I. We create $\prod_{\mathcal{U}} M_i$ as follows:

- If S is a sort, form $\prod_{\mathcal{U}} S(M_i)$.
- Define functions coordinatewise on sorts.
- For a relation R and $\overline{a}$ from an appropriate sort, let

$$R(\overline{a}) = \lim_{i \to \mathcal{U}} R^{M_i}(\overline{a}_i).$$

Model theoretic use of ultraproducts

Theorem

For a class of $\mathcal{L}$-structures $\mathcal{C}$, TFAE

- *$\mathcal{C}$ is an elementary class i.e. all $\mathcal{L}$-structures satisfying some set of sentences.*
- *$\mathcal{C}$ is closed under isomorphisms, ultraproducts and elementary submodels.*
- *$\mathcal{C}$ is closed under isomorphisms, ultraproducts and ultraroots.*

Background on correspondences

Fix a tracial von Neumann algebra (M, τ).

- A representation of an operator algebra A on a Hilbert space H is a *-homomorphism from A to $B(H)$.
- An M-correspondence is a Hilbert space H together with commuting normal representations π_M and $\pi_{M^{op}}$, the left and right actions.
- If H is an M-correspondence, $\xi \in H$ is K-bounded if for all $c \in M_+$,

$$\langle c\xi, \xi \rangle \leq K\tau(c) \text{ and } \langle \xi c, \xi \rangle \leq K\tau(c).$$

- A correspondence is called cyclic if it is singly generated by a bounded vector.
- Fact: Every correspondence is the direct sum of cyclic correspondences. In fact, the bounded vectors are dense in any correspondence.

The language of correspondences

Fix tracial von Neumann algebras M. The language $\mathcal{L}$ of an M-correspondence H and its intended interpretation are:

- for each $K \in \mathbb{N}$, there will be a sort S_K and $S_K(H)$ will be the set of K-bounded vectors. The metric will be induced by the inner product on H;
- for $K < L$ there will be an isometry from S_K to S_L which for a given correspondence will be interpreted as the inclusion map;
- $+$ will be defined on all pairs of sorts and will be interpreted standardly as the restriction of addition; and,
- there will be unary functions for each $c \in M$ which implement the left and right actions.

We call the structure in this language associated with H, the dissection of H, $D(H)$.

Ultraproducts of correspondences

1. Fix M-correspondences H_i for $i \in I$ and an ultrafilter $\mathcal{U}$ on I. We can form the ultraproduct in two ways:
2. We could take the ultraproducts of the dissections. This amounts to forming $S_K(H_i)$ for each K and i and let H, the ultraproduct, be the closure of

$$\bigcup_K \left(\prod_{\mathcal{U}} S_K(H_i) \right)$$

in $\prod_{\mathcal{U}} H_i$.

3. Alternatively, we could take those $\xi \in \prod_{\mathcal{U}} H_i$ at which the left and right actions at ξ are continuous i.e. $L_\xi : M \to \prod_{\mathcal{U}} H_i$ and $R_\xi : M^{op} \to \prod_{\mathcal{U}} H_i$ are bounded.

Theorem

1. *If M is a tracial von Neumann algebra then the class of M-correspondences forms an elementary class.*
2. *The theory of M-correspondences is*
   1. *stable,*
   2. *classifiable, and*
   3. *has a model companion.*

The lazy explanation

- Recall that every M-correspondence is a direct sum of cyclic correspondences.
- Lazily, there are only boundedly many cyclic correspondences because they are bounded in size.
- Similarly, a lazy description of the model companion is the theory of

$$\overline{\bigoplus_{H \in \mathcal{P}} H^{\oplus \omega}}$$

where $\mathcal{P}$ is the set of all cyclic correspondences.

A deeper explanation

Definition

An element of M of the form a^*a is called positive.

1. A linear map is positive if it sends positive elements to positive elements.
2. $\varphi : M \to M$ is completely positive (c.p.) if φ_n is positive for all n where $\varphi_n : M \otimes M_n(C) \to : M \otimes M_n(C)$ such that

$$\varphi_n(a \otimes b) = \varphi(a) \otimes b.$$

A deeper explanation, cont'd

- Fix a c.p. map $\varphi : M \to M$ and consider the form on $M \otimes M$ determined by

$$\langle a \otimes b, c \otimes d \rangle_\varphi = \tau_M(\varphi(c^* a) d^* b).$$

- The completion of $M \otimes M / \langle \cdot, \cdot \rangle_\varphi$ with the left and right action induced from the tensor product is a cyclic correspondence with cyclic vector $1 \otimes 1$.
- In fact, all cyclic correspondences have this form and every bounded vector in a correspondence gives rise to a c.p. map from M to M.
- $L^2(M, \tau)$ corresponds to the identity function and is the trivial correspondence.

Some open questions

- Independence and orthogonality are well understood in a formal sense in theories of correspondences.
- Is there a good notion of minimal cyclic correspondence and if so, what do the associated c.p. maps look like?
- Say that a correspondence H is ample if whenever K is a cyclic correspondence and K embeds into $H^{\mathcal{U}}$ then $K^{\oplus\omega}$ embeds into $H^{\mathcal{U}}$. Equivalently the algebraic closure of the empty set in the theory of H is 0.
- Fact: If H is ample then the theory of H has quantifier elimination.
- Is every theory of correspondences ample? In particular, for a given (M, τ), is $L^2(M, \tau)$ ample?
- A question which is not open is whether the central vectors form a definable set in the theory of M-correspondences. They do.

Thank you.
Happy birthday Chris!