More on Generic Structures

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A look at the setting

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1. an *amalgamation class* if $(K, \leq)$ satisfies the amalgamation and joint embedding properties.
2. a *smooth class* if for all $A \in K$, there is a set of universal formulas $p^A(\overline{x})$ such that for any $B \in K$ with $A \subseteq B$

$$A \leq B \iff B \models \varphi(\overline{a}) \text{ for all } \varphi \in p^A$$

Definition

Let M be an L structure whose finite substructures are *cofinal* in K, i.e. for any finite $A \subseteq M$, there is some $B \in K$ with $A \subseteq B \subseteq M$. We let the class of such structures be denoted by $\overline{K}$.

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Given $M \in \overline{K}$ and $A \in K$ with $A \subseteq M$, we say that A is *strong* in M, denoted by $A \leq M$, if for all finite $B \subseteq M$ with $A \subseteq B$ and $B \in K$, we have that $A \leq B$.

Definition

Given an amalgmation class $(K, \leq)$ we say that M is a *generic* structure for $(K, \leq)$ if

1. M is the union of an ω-chain $A_0 \leq A_1 \leq \ldots$ with each $A_i \in K$.
2. If $A, B \in K$ with $A \leq B$ and $A \leq M$, then there is $B' \leq M$ such that $B \cong_A B'$.
3. If $A \in K$, then there is some embedding $f : A \to M$ such that $f(A) \leq M$.

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Note that a $(K, \leq)$ generic is in $\overline{K}$.

Theorem (Fraïssé)

Let $(K, \leq)$ be an amalgamation class. A $(K, \leq)$ generic M exists and is unique up to isomorphism.

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Definition

A smooth class $(K, \leq)$ is called a *Fraïssé-Hrushovski class* (or Fraïssé class for short) if it has the hereditary property and satisfies the *intersection property*, i.e. if $A, C \in K$ with $A \leq C$ and $B \subseteq C$, then $A \cap B \leq B$.

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3. Hrushovski constructions

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Theorem (Kueker, Laskowski)

Let $(K, \leq)$ be a Fraïssé class, M be the $(K, \leq)$ generic.

1. *M is saturated if and only if M is weakly saturated.*
2. *If the notion of $\leq$ is captured by a single formula, then M is prime.*

Definition

Let $(K, \leq)$ be a Fraïssé class.

1. Given $A, B \in K$ with $A \subseteq B$, we say that (A, B) is a *minimal pair* if $A \nleq B$ but $A \leq C$ for all $A \subseteq C \subsetneq B$

Definition

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2. Given $M \in \overline{K}$, $X \subseteq M$ we say that X *is closed in* M if for any finite $A \subseteq X$, minimal pair (A, B) with $B \subseteq M$, we have that $B \subseteq X$. The smallest closed set containing X in M is called the *closure of* X *in* M (denoted $\mathrm{cl}_M(X)$).

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Fact

1. *cl_M is a closure operation.*
2. *For a finite set $A \leq M$ if and only if $cl_M(A) = A$.*

Lemma

Let $(K, \leq)$ be a Fraïssé class and M be the $(K, \leq)$ generic. If $Th(M)$ admits a prime model N, then N has finite closures, i.e. for any finite $A \subseteq N$, $cl_N(A)$ is finite.

Lemma

Let $(K, \leq)$ *be a Fraïssé class and* M *be the* $(K, \leq)$ *generic. If* $Th(M)$ *admits a prime model* N*, then* N *has finite closures, i.e. for any finite* $A \subseteq N$*,* $cl_N(A)$ *is finite.*

Theorem

Let $(K, \leq)$ *be a Fraïssé class and assume that the* $(K, \leq)$ *generic* M *is* $\aleph_0$*-saturated. Then there is a* $K_0 \subseteq K$ *such that*

1. *For each* $A \in K$*, there is some* $B \in K_0$ *with* $A \subseteq B$*.*
2. $(K_0, \leq)$ *is an amalgamation class.*
3. *The* $(K_0, \leq)$ *generic is isomorphic to* N*.*

Question

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2. When can we say something more concrete about K_0?

Standard Axioms

Definition

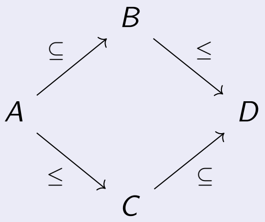
Let $(K, \leq)$ be a Fraïssé class. The *standard axioms for* $(K, \leq)$ is the set of axioms S_K that guarantee for $M \models S_K$,

1. $M \in \overline{K}$
2. If $A, B \in K$ with $A \subseteq M$, $A \leq B$, then there exists $B' \subseteq M$ such that $B' \cong_A B$

Full amalgamation

Definition

Let $(K, \leq)$ be a Fraïssé class.We say that $(K, \leq)$ has *full amalgamation* if for all $A, B, C \in K$ with $A \leq C$, $A \subseteq B$, there is some $D \in K$ such that the following diagram commutes



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1. $M \models S_K$ *if and only if* $(K, \leq)$ *has full amalgamation.*

Theorem

Let $(K, \leq)$ *be a Fraïssé class,* M *be the* $(K, \leq)$ *generic.*

1. $M \models S_K$ *if and only if* $(K, \leq)$ *has full amalgamation.*
2. *If* $Th(M)$ *admits finite closures, then any countable model is atomically prime.*

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For any finite L-structure A define $\delta(A) = |A| - \sum_{E \in L} \overline{\alpha}(E)|E^A|$.

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Definition

Let $K_{\overline{\alpha}} = \{A : A \text{ is a finite } L\text{-structure and } \delta(A') \geq 0 \text{ for all } A' \subseteq A\}$

Definition

Let $A, B \in K$. We say that A *is strong in* B (denoted by $A \leq B$) if $A \subseteq B$ and for all $A \subseteq B' \subseteq B$, we have that $\delta(B') \geq \delta(A)$.

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We will focus our attention on Fraïssé classes $(K, \leq)$ where $K \subseteq K_{\overline{\alpha}}$ with the notion of strong being inherited.

Guiding Example: Baldwin-Shi hypergraphs

Consider $\overline{\alpha} : L \to (0, 1]$ and $K = K_{\overline{\alpha}}$. Denote the theory of the $(K_{\overline{\alpha}}, \leq)$ generic by $S_{\overline{\alpha}}$. Note $S_{K_{\overline{\alpha}}} \subseteq S_{\overline{\alpha}}$.

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4. *If $N, M \models S_{\overline{\alpha}}$, $N \subseteq M$ and N is algebraically closed, then $N \preccurlyeq M$ [G.]*
5. *$M \models S_{\overline{\alpha}}$ is atomic if and only if for any finite $A \subseteq M$ there is a finite $B \subseteq M$ with $\delta(B) = 0$ and $A \subseteq B$.*

Laskowski's approach allows for $2, 3, 4, 5$ to be obtained by 1.

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Definition

We say that $(K, \leq)$ *omits embeddings* if for any $A \leq B \in K$ and any finite $\Phi \subset K$ with $B \subseteq C$ and $B \not\leq C$ for all $C \in \Phi$, for any $\mu > 0$, there is a $D \supseteq B$, $D \in K$ such that

1. $0 \leq \delta(D/A) < \mu$
2. $A \leq D$
3. No $C \in \Phi$ isomorphically embeds into D over B

We also write $(K, \leq)$ has the *OEP*.

The quantifier elimination result may be stated as

Theorem

If $(K, \leq)$ has the OEP, then S_K admits the relevant level of quantifier elimination, is complete,....

Consider $\overline{\alpha}' : L \to \omega - \{0\}$.

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Lemma (G.)

$(K_{\overline{\alpha}'}, \leq)$ *has the OEP and hence admits....*

Theorem (?)

Let L^* be an infinite relational language where each $E \in L$ has arity at least 2. Let $\overline{\beta} : L^* \to \omega - \{0\}$ and $\delta, K_{\overline{\beta}}$ be defined canonically. The $(K_{\overline{\beta}}, \leq)$ generic M is $\aleph_0$-saturated

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Proof:...

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3. *The countable models form an elementary chain $M_0 \preccurlyeq M_1 \preccurlyeq \ldots \preccurlyeq M_\omega$ with M_0 atomic and M_ω the generic.[G.]*

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The key to proving these theorems is the existence of finite structures in $K_{\overline{\alpha}}$ with even more specific properties.

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We say that the the class $(K, \leq)$ has the *essential minimal pair property*, if for any $A \in K$ with $\delta(A) > 0$ there are infinitely many non-isomorphic $D \in K$ such that

1. (A, D) is an essential minimal pair that satisfies $\delta(D/A) = -1/c$
2. Given any $C \leq A$ with $\delta(A/C) \geq 1/c$ and the unique non-negative integer k satisfying $-k\delta(D/A) = \delta(A/C)$, the free join of k-many isomorphic copies of D over A, $\oplus_{i<k} D_i$, lies in K. We also write $(K, \leq)$ has the *EMP*

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Theorem

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Example

1. Baldwin-Shi hypergraphs where $\overline{\alpha}$ is rational valued.

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A version of the EMP exists for non-rational valued $\overline{\alpha}$. In the case of $\overline{\alpha}$ takes values in $(0, 1]$ the EMP is reflected in the triviality of forking.

Full Circle: 0-1 laws and pseudofiniteness

Let L contain a single binary relation. If α is irrational, then there is a corresponding $0-1$ law for $S_{\overline{\alpha}}$. If α is rational, then $S_{\overline{\alpha}}$ is pseudofinite.

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There is a rat in separate. ___________,*Chris Laskowski*

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Thank you!