Dp-minimal ordered Abelian groups revisited

John Goodrick

Universidad de los Andes
Bogotá, Colombia

Conference for Chris Laskowski's birthday
June 22, 2019

Motivation

We would like to generalize some results about stable algebraic structures to the NIP (a.k.a. "geometric") setting.

Theorem:

1. (Reineke) Any strongly minimal group is Abelian.
2. (Macintyre) Any strongly minimal field (or even ω-stable field) is algebraically closed.

Question: If we replace "strongly minimal" by "dp-minimal," what can we say?

(More or less complete answer for fields by Johnson, partial answer for groups by Kaplan, Levi, Simon.)

Motivation

We would like to generalize some results about stable algebraic structures to the NIP (a.k.a. "geometric") setting.

Theorem:

1. (Reineke) Any strongly minimal group is Abelian.
2. (Macintyre) Any strongly minimal field (or even ω-stable field) is algebraically closed.

Question: If we replace "strongly minimal" by "dp-minimal," what can we say?

(More or less complete answer for fields by Johnson, partial answer for groups by Kaplan, Levi, Simon.)

Motivation

We would like to generalize some results about stable algebraic structures to the NIP (a.k.a. "geometric") setting.

Theorem:

1. (Reineke) Any strongly minimal group is Abelian.
2. (Macintyre) Any strongly minimal field (or even ω-stable field) is algebraically closed.

Question: If we replace "strongly minimal" by "dp-minimal," what can we say?

(More or less complete answer for fields by Johnson, partial answer for groups by Kaplan, Levi, Simon.)

Motivation

We would like to generalize some results about stable algebraic structures to the NIP (a.k.a. "geometric") setting.

Theorem:

1. (Reineke) Any strongly minimal group is Abelian.
2. (Macintyre) Any strongly minimal field (or even ω-stable field) is algebraically closed.

Question: If we replace "strongly minimal" by "dp-minimal," what can we say?

(More or less complete answer for fields by Johnson, partial answer for groups by Kaplan, Levi, Simon.)

Motivation, Part 2

Dp-minimality generalizes strong minimality, but allows orderings and valuations.

Examples of unstable dp-minimal structures:

- Any o-minimal structure.
- $(\mathbb{Z}, +, <)$ (Presburger arithmetic).
- The field of p-adic numbers.

Question: All of the above have nice cell decomposition and "tame topology" (except perhaps $(\mathbb{Z}, +, <)$). Do definable sets in (unstable) dp-minimal algebraic structures generally have nice topological properties?

Motivation, Part 2

Dp-minimality generalizes strong minimality, but allows orderings and valuations.

Examples of unstable dp-minimal structures:

- Any o-minimal structure.
- $(\mathbb{Z}, +, <)$ (Presburger arithmetic).
- The field of p-adic numbers.

Question: All of the above have nice cell decomposition and "tame topology" (except perhaps $(\mathbb{Z}, +, <)$). Do definable sets in (unstable) dp-minimal algebraic structures generally have nice topological properties?

Motivation, Part 2

Dp-minimality generalizes strong minimality, but allows orderings and valuations.

Examples of unstable dp-minimal structures:

- Any o-minimal structure.
- $(\mathbb{Z}, +, <)$ (Presburger arithmetic).
- The field of *p*-adic numbers.

Question: All of the above have nice cell decomposition and "tame topology" (except perhaps $(\mathbb{Z}, +, <)$). Do definable sets in (unstable) dp-minimal algebraic structures generally have nice topological properties?

Outline of Talk

Outline of Talk

1. Dp-minimality: context, general facts
2. Dp-minimal (Abelian) groups
3. Dp-minimal ordered (Abelian) groups
4. Discretely ordered Abelian groups
5. Finite dp-rank ordered Abelian groups

Outline of Talk

1. Dp-minimality: context, general facts
2. Dp-minimal (Abelian) groups
3. Dp-minimal ordered (Abelian) groups
4. Discretely ordered Abelian groups
5. Finite dp-rank ordered Abelian groups

Outline of Talk

1. Dp-minimality: context, general facts
2. Dp-minimal (Abelian) groups
3. Dp-minimal ordered (Abelian) groups
4. Discretely ordered Abelian groups
5. Finite dp-rank ordered Abelian groups

Outline of Talk

1. Dp-minimality: context, general facts
2. Dp-minimal (Abelian) groups
3. Dp-minimal ordered (Abelian) groups
4. Discretely ordered Abelian groups
5. Finite dp-rank ordered Abelian groups

Dp-minimality: the definition

Fix a complete theory T and $\mathscr{M}$ a $|T|^+$-saturated model.

An **ict-pattern of depth κ in** $p(x;a)$ is an array of formulas $\{\varphi_{i,j}(x;b_{i,j}) : i<\kappa, j<\omega\}$ such that for every function $\eta:\kappa\to\omega$, the partial type

$$p(x;a)\cup\{\varphi_{i,\eta(i)}(x,b_{i,\eta(i)}) : i<\kappa\}$$

$$\cup\{\neg\varphi_{i,j}(x,b_{i,j}) : i<\kappa, j\neq\eta(i)\}$$

is consistent.[1]

The **dp-rank** of a partial type $p(x;a)$ is the maximum κ such that there is an ict-pattern of depth κ in $p(x;a)$, or "κ^-" if there are ict-patterns of depth λ for all $\lambda<\kappa$ but not of depth κ.

T is **dp-minimal** if the dp-rank of $x=x$ (in a single variable) is 1.

[1]Here $x, a, b_{i,j}$ may be finite tuples and $a, b_{ij}\in M$.

Dp-minimality: the definition

Fix a complete theory T and $\mathscr{M}$ a $|T|^+$-saturated model.

An **ict-pattern of depth κ in** $p(x;a)$ is an array of formulas $\{\varphi_{i,j}(x;b_{i,j}) : i<\kappa, j<\omega\}$ such that for every function $\eta:\kappa\to\omega$, the partial type

$$p(x;a)\cup\{\varphi_{i,\eta(i)}(x,b_{i,\eta(i)}) : i<\kappa\}$$

$$\cup\{\neg\varphi_{i,j}(x,b_{i,j}) : i<\kappa, j\neq\eta(i)\}$$

is consistent.[1]

The **dp-rank** of a partial type $p(x;a)$ is the maximum κ such that there is an ict-pattern of depth κ in $p(x;a)$, or "κ^-" if there are ict-patterns of depth λ for all $\lambda<\kappa$ but not of depth κ.

T is **dp-minimal** if the dp-rank of $x=x$ (in a single variable) is 1.

[1] Here $x, a, b_{i,j}$ may be finite tuples and $a, b_{ij}\in M$.

Dp-rank and weight

Dp-rank in NIP theories plays a role similar to weight in stable theories.

Theorem: (Adler) If p is a complete type in a stable theory, then dp-rk(p) is the maximum weight of any extension of p.

So dp-minimal stable theories are theories in which every 1-type has weight 1.

Theorem: (Kaplan, Onshuus, Usvyatsov) dp-rk(p) $\geq \kappa$ if and only if there is a set A, mutually A-indiscernible sequences $\langle \mathscr{I}_i : i < \kappa \rangle$, and a realization c of p such that for every $i < \kappa$, $\mathscr{I}_i$ is not Ac-indiscernible.

Dp-rank and weight

Dp-rank in NIP theories plays a role similar to weight in stable theories.

Theorem: (Adler) If p is a complete type in a stable theory, then dp-rk(p) is the maximum weight of any extension of p.

So dp-minimal stable theories are theories in which every 1-type has weight 1.

Theorem: (Kaplan, Onshuus, Usvyatsov) dp-rk(p) $\geq \kappa$ if and only if there is a set A, mutually A-indiscernible sequences $\langle \mathscr{I}_i : i < \kappa \rangle$, and a realization c of p such that for every $i < \kappa$, $\mathscr{I}_i$ is not Ac-indiscernible.

Dp-rank and weight

Dp-rank in NIP theories plays a role similar to weight in stable theories.

Theorem: (Adler) If p is a complete type in a stable theory, then dp-rk(p) is the maximum weight of any extension of p.

So dp-minimal stable theories are theories in which every 1-type has weight 1.

Theorem: (Kaplan, Onshuus, Usvyatsov) dp-rk(p) $\geq \kappa$ if and only if there is a set A, mutually A-indiscernible sequences $\langle \mathscr{I}_i : i < \kappa \rangle$, and a realization c of p such that for every $i < \kappa$, $\mathscr{I}_i$ is not Ac-indiscernible.

Dp-rank and weight

Dp-rank in NIP theories plays a role similar to weight in stable theories.

Theorem: (Adler) If p is a complete type in a stable theory, then dp-rk(p) is the maximum weight of any extension of p.

So dp-minimal stable theories are theories in which every 1-type has weight 1.

Theorem: (Kaplan, Onshuus, Usvyatsov) dp-rk(p) $\geq \kappa$ if and only if there is a set A, mutually A-indiscernible sequences $\langle \mathscr{I}_i : i < \kappa \rangle$, and a realization c of p such that for every $i < \kappa$, $\mathscr{I}_i$ is not Ac-indiscernible.

Basic facts about dp-minimality

- A formula has dp-rank ≥ 1 if and only if it is nonalgebraic.
- If $T \vdash \forall x \varphi(x; a) \to \psi(x; b)$, then dp-rk$(\varphi(x; a)) \leq$ dp-rk$(\psi(x; b))$.
- *Subadditivity:* dp-rk(tp$(ab/B) \leq$ dp-rk(tp$(a/B)) +$ dp-rk(tp$(b/aB))$.
- T is NIP (geometric) if and only if every formula has bounded dp-rank.

Related Notions

$$\begin{array}{ccccccc}
\text{inp-minimal} & \Rightarrow & \text{finite burden} & \Rightarrow & \text{strong} & \Rightarrow & \text{NTP}_2 \\
\Uparrow & & \Uparrow & & \Uparrow & & \Uparrow \\
\text{dp-minimal} & \Rightarrow & \text{finite dp-rank} & \Rightarrow & \text{strong} + \text{NIP} & \Rightarrow & \text{NIP} \\
\Uparrow & & \Uparrow & & \Uparrow & & \Uparrow \\
\text{weakly minimal} & \Rightarrow & \text{finite weight} & \Rightarrow & \text{strong} + \text{stable} & \Rightarrow & \text{stable}
\end{array}$$

"Strong + NIP" is equivalent to: there is no ict-pattern of depth $\aleph_0$. (But there may be ict-patterns of arbitrarily large finite depth!)

Note: there are stable, non superstable theories of finite weight.

Related Notions

$$
\begin{array}{ccccccc}
\text{inp-minimal} & \Rightarrow & \text{finite burden} & \Rightarrow & \text{strong} & \Rightarrow & NTP_2 \\
\Uparrow & & \Uparrow & & \Uparrow & & \Uparrow \\
\text{dp-minimal} & \Rightarrow & \text{finite dp-rank} & \Rightarrow & \text{strong} + \text{NIP} & \Rightarrow & \text{NIP} \\
\Uparrow & & \Uparrow & & \Uparrow & & \Uparrow \\
\text{weakly minimal} & \Rightarrow & \text{finite weight} & \Rightarrow & \text{strong} + \text{stable} & \Rightarrow & \text{stable}
\end{array}
$$

"Strong + NIP" is equivalent to: there is no ict-pattern of depth $\aleph_0$. (But there may be ict-patterns of arbitrarily large finite depth!)

Note: there are stable, non superstable theories of finite weight.

Related Notions

$$
\begin{array}{ccccccc}
\text{inp-minimal} & \Rightarrow & \text{finite burden} & \Rightarrow & \text{strong} & \Rightarrow & \text{NTP}_2 \\
\Uparrow & & \Uparrow & & \Uparrow & & \Uparrow \\
\text{dp-minimal} & \Rightarrow & \text{finite dp-rank} & \Rightarrow & \text{strong} + \text{NIP} & \Rightarrow & \text{NIP} \\
\Uparrow & & \Uparrow & & \Uparrow & & \Uparrow \\
\text{weakly minimal} & \Rightarrow & \text{finite weight} & \Rightarrow & \text{strong} + \text{stable} & \Rightarrow & \text{stable}
\end{array}
$$

“Strong + NIP” is equivalent to: there is no ict-pattern of depth $\aleph_0$. (But there may be ict-patterns of arbitrarily large finite depth!)

Note: there are stable, non superstable theories of finite weight.

Dp-minimal orders and trees

Theorem: (Simon) Any complete theory of a linear ordering is dp-minimal.

In fact, any complete theory of a "colored linear ordering" (linear order expanded by arbitrary unary predicates) is dp-minimal.

Theorem: (Simon) Any complete theory of a tree is dp-minimal. (A "tree" is a partial order $(T, \leq)$ such that the set of elements below any $a \in M$ is linearly ordered, and for any $a, b \in T$ there is a c below a and b.)

Dp-minimal orders and trees

Theorem: (Simon) Any complete theory of a linear ordering is dp-minimal.

In fact, any complete theory of a "colored linear ordering" (linear order expanded by arbitrary unary predicates) is dp-minimal.

Theorem: (Simon) Any complete theory of a tree is dp-minimal. (A "tree" is a partial order $(T, \leq)$ such that the set of elements below any $a \in M$ is linearly ordered, and for any $a, b \in T$ there is a c below a and b.)

Dp-minimal orders and trees

Theorem: (Simon) Any complete theory of a linear ordering is dp-minimal.

In fact, any complete theory of a "colored linear ordering" (linear order expanded by arbitrary unary predicates) is dp-minimal.

Theorem: (Simon) Any complete theory of a tree is dp-minimal. (A "tree" is a partial order $(T, \leq)$ such that the set of elements below any $a \in M$ is linearly ordered, and for any $a, b \in T$ there is a c below a and b.)

Dp-minimal groups

Question: Which groups $(G, \cdot)$ in the pure group language are dp-minimal? What about the Abelian case?

Easy Lemma: If H, K are definable normal subgroups of G such that $[H : H \cap K] = \infty$ and $[K : H \cap K] = \infty$, then $(G, \cdot)$ is **not** dp-minimal.

Lemma: (Simon) If G is a dp-minimal group (or even inp-minimal), then there is a definable normal Abelian $H \leq G$ such that G/H has finite exponent.

Theorem: (Halevi-Hasson) An **Abelian** group $(G, +)$ is dp-minimal if and only if there is at most one prime p such that $[G : pG] = \infty$.

Dp-minimal groups

Question: Which groups $(G, \cdot)$ in the pure group language are dp-minimal? What about the Abelian case?

Easy Lemma: If H, K are definable normal subgroups of G such that $[H : H \cap K] = \infty$ and $[K : H \cap K] = \infty$, then $(G, \cdot)$ is **not** dp-minimal.

Lemma: (Simon) If G is a dp-minimal group (or even inp-minimal), then there is a definable normal Abelian $H \leq G$ such that G/H has finite exponent.

Theorem: (Halevi-Hasson) An **Abelian** group $(G, +)$ is dp-minimal if and only if there is at most one prime p such that $[G : pG] = \infty$.

Dp-minimal groups

Question: Which groups $(G, \cdot)$ in the pure group language are dp-minimal? What about the Abelian case?

Easy Lemma: If H, K are definable normal subgroups of G such that $[H : H \cap K] = \infty$ and $[K : H \cap K] = \infty$, then $(G, \cdot)$ is **not** dp-minimal.

Lemma: (Simon) If G is a dp-minimal group (or even inp-minimal), then there is a definable normal Abelian $H \leq G$ such that G/H has finite exponent.

Theorem: (Halevi-Hasson) An **Abelian** group $(G, +)$ is dp-minimal if and only if there is at most one prime p such that $[G : pG] = \infty$.

Dp-minimal groups

Question: Which groups $(G, \cdot)$ in the pure group language are dp-minimal? What about the Abelian case?

Easy Lemma: If H, K are definable normal subgroups of G such that $[H : H \cap K] = \infty$ and $[K : H \cap K] = \infty$, then $(G, \cdot)$ is **not** dp-minimal.

Lemma: (Simon) If G is a dp-minimal group (or even inp-minimal), then there is a definable normal Abelian $H \leq G$ such that G/H has finite exponent.

Theorem: (Halevi-Hasson) An **Abelian** group $(G, +)$ is dp-minimal if and only if there is at most one prime p such that $[G : pG] = \infty$.

Dp-minimal expansions of $(\mathbb{Z}, +)$

The complete theory of $(\mathbb{Z}, +)$ is stable and weakly minimal, hence dp-minimal.

Question: Which expansions of $(\mathbb{Z}, +)$ are dp-minimal?

Fact: (Conant-Pillay) Any proper expansion of $(\mathbb{Z}, +)$ which is stable has infinite dp-rank.

Examples The following structures are all dp-minimal:

- $(\mathbb{Z}, <, +)$;
- (Alouf-d'Elbée) $(\mathbb{Z}, |_p, +)$, where p is prime and $x|_p y$ means that for every $k \in \mathbb{N}$, if $p^k | x$ then p^k divides y;
- (Tran-Walsberg) Expansions of $(\mathbb{Z}, +)$ by "cyclic orderings."

Dp-minimal expansions of $(\mathbb{Z}, +)$

The complete theory of $(\mathbb{Z}, +)$ is stable and weakly minimal, hence dp-minimal.

Question: Which expansions of $(\mathbb{Z}, +)$ are dp-minimal?

Fact: (Conant-Pillay) Any proper expansion of $(\mathbb{Z}, +)$ which is stable has infinite dp-rank.

Examples The following structures are all dp-minimal:

- $(\mathbb{Z}, <, +)$;
- (Alouf-d'Elbée) $(\mathbb{Z}, |_p, +)$, where p is prime and $x|_p y$ means that for every $k \in \mathbb{N}$, if $p^k|x$ then p^k divides y;
- (Tran-Walsberg) Expansions of $(\mathbb{Z}, +)$ by "cyclic orderings."

Dp-minimal expansions of $(\mathbb{Z}, +)$

The complete theory of $(\mathbb{Z}, +)$ is stable and weakly minimal, hence dp-minimal.

Question: Which expansions of $(\mathbb{Z}, +)$ are dp-minimal?

Fact: (Conant-Pillay) Any proper expansion of $(\mathbb{Z}, +)$ which is stable has infinite dp-rank.

Examples The following structures are all dp-minimal:

- $(\mathbb{Z}, <, +)$;
- (Alouf-d'Elbée) $(\mathbb{Z}, |_p, +)$, where p is prime and $x|_p y$ means that for every $k \in \mathbb{N}$, if $p^k|x$ then p^k divides y;
- (Tran-Walsberg) Expansions of $(\mathbb{Z}, +)$ by "cyclic orderings."

Dp-minimal fields

Some examples of dp-minimal fields are:

- Any algebraically closed field (which is strongly minimal);
- The real closed field $(\mathbb{R}, <, +, \cdot)$ (which is o-minimal);
- The p-adic field $\mathbb{Q}_p$ (Dolich, G., Lippel) or any finite extension thereof.

Theorem: (Will Johnson) If K is dp-minimal, sufficiently saturated, and not algebraically closed, then K admits a nontrivial defectless Henselian valuation $v : K \to \Gamma$ such that:

- For every prime p, $\Gamma/p\Gamma$ is finite;
- the residue field k is algebraically closed, real closed, or a finite extension of a p-adic field; and
- if $\mathrm{char}(k) = p$, then $[v(-p), v(p)] \subseteq p\Gamma$.

Dp-minimal fields

Some examples of dp-minimal fields are:

- Any algebraically closed field (which is strongly minimal);
- The real closed field $(\mathbb{R}, <, +, \cdot)$ (which is o-minimal);
- The p-adic field $\mathbb{Q}_p$ (Dolich, G., Lippel) or any finite extension thereof.

Theorem: (Will Johnson) If K is dp-minimal, sufficiently saturated, and not algebraically closed, then K admits a nontrivial defectless Henselian valuation $v : K \to \Gamma$ such that:

- For every prime p, $\Gamma/p\Gamma$ is finite;
- the residue field k is algebraically closed, real closed, or a finite extension of a p-adic field; and
- if $\text{char}(k) = p$, then $[v(-p), v(p)] \subseteq p\Gamma$.

Dp-minimal fields

Some examples of dp-minimal fields are:

- Any algebraically closed field (which is strongly minimal);
- The real closed field $(\mathbb{R}, <, +, \cdot)$ (which is o-minimal);
- The p-adic field $\mathbb{Q}_p$ (Dolich, G., Lippel) or any finite extension thereof.

Theorem: (Will Johnson) If K is dp-minimal, sufficiently saturated, and not algebraically closed, then K admits a nontrivial defectless Henselian valuation $v : K \to \Gamma$ such that:

- For every prime p, $\Gamma/p\Gamma$ is finite;
- the residue field k is algebraically closed, real closed, or a finite extension of a p-adic field; and
- if $\mathrm{char}(k) = p$, then $[v(-p), v(p)] \subseteq p\Gamma$.

Dp-minimal fields

Some examples of dp-minimal fields are:

- Any algebraically closed field (which is strongly minimal);
- The real closed field $(\mathbb{R}, <, +, \cdot)$ (which is o-minimal);
- The p-adic field $\mathbb{Q}_p$ (Dolich, G., Lippel) or any finite extension thereof.

Theorem: (Will Johnson) If K is dp-minimal, sufficiently saturated, and not algebraically closed, then K admits a nontrivial defectless Henselian valuation $v : K \to \Gamma$ such that:

- For every prime p, $\Gamma/p\Gamma$ is finite;
- the residue field k is algebraically closed, real closed, or a finite extension of a p-adic field; and
- if $\mathrm{char}(k) = p$, then $[v(-p), v(p)] \subseteq p\Gamma$.

Dp-minimal fields

Some examples of dp-minimal fields are:

- Any algebraically closed field (which is strongly minimal);
- The real closed field $(\mathbb{R}, <, +, \cdot)$ (which is o-minimal);
- The p-adic field $\mathbb{Q}_p$ (Dolich, G., Lippel) or any finite extension thereof.

Theorem: (Will Johnson) If K is dp-minimal, sufficiently saturated, and not algebraically closed, then K admits a nontrivial defectless Henselian valuation $v : K \to \Gamma$ such that:

- For every prime p, $\Gamma/p\Gamma$ is finite;
- the residue field k is algebraically closed, real closed, or a finite extension of a p-adic field; and
- if $\text{char}(k) = p$, then $[v(-p), v(p)] \subseteq p\Gamma$.

“Shelah’s conjecture” for NIP fields

Shelah’s conjecture: Any NIP (or geometric) infinite field is either algebraically closed, real closed, or admits a nontrivial Henselian valuation (which is not necessarily definable!).

Henselianity conjecture: Any definable valuation in an NIP field is Henselian.

These conjectures have recently been confirmed for positive characteristic fields of finite dp-rank by Will Johnson.

“Shelah’s conjecture” for NIP fields

Shelah’s conjecture: Any NIP (or geometric) infinite field is either algebraically closed, real closed, or admits a nontrivial Henselian valuation (which is not necessarily definable!).

Henselianity conjecture: Any definable valuation in an NIP field is Henselian.

These conjectures have recently been confirmed for positive characteristic fields of finite dp-rank by Will Johnson.

"Shelah's conjecture" for NIP fields

Shelah's conjecture: Any NIP (or geometric) infinite field is either algebraically closed, real closed, or admits a nontrivial Henselian valuation (which is not necessarily definable!).

Henselianity conjecture: Any definable valuation in an NIP field is Henselian.

These conjectures have recently been confirmed for positive characteristic fields of finite dp-rank by Will Johnson.

Preliminaries on ordered groups

Recall: an **ordered Abelian group** is an Abelian group $(G, <, +)$ with a translation-invariant ordering $<$.

• (Levi) An Abelian group admits an ordering if and only if it is torsion-free.

• A left-invariant ordering on $(G, \cdot)$ is either **discrete** (if there is a least positive element) or **dense**.

Fun Fact: The free group with 2 generators admits a bi-invariant ordering.

Preliminaries on ordered groups

Recall: an **ordered Abelian group** is an Abelian group $(G, <, +)$ with a translation-invariant ordering $<$.

• (Levi) An Abelian group admits an ordering if and only if it is torsion-free.

• A left-invariant ordering on $(G, \cdot)$ is either **discrete** (if there is a least positive element) or **dense**.

Fun Fact: The free group with 2 generators admits a bi-invariant ordering.

Preliminaries on ordered groups

Recall: an **ordered Abelian group** is an Abelian group $(G, <, +)$ with a translation-invariant ordering $<$.

• (Levi) An Abelian group admits an ordering if and only if it is torsion-free.

• A left-invariant ordering on $(G, \cdot)$ is either **discrete** (if there is a least positive element) or **dense**.

Fun Fact: The free group with 2 generators admits a bi-invariant ordering.

Preliminaries on ordered groups

Recall: an **ordered Abelian group** is an Abelian group $(G, <, +)$ with a translation-invariant ordering $<$.

• (Levi) An Abelian group admits an ordering if and only if it is torsion-free.

• A left-invariant ordering on $(G, \cdot)$ is either **discrete** (if there is a least positive element) or **dense**.

Fun Fact: The free group with 2 generators admits a bi-invariant ordering.

Abelianity and left-invariant orders

Theorem: (Simon) Every bi-ordered dp-minimal group is Abelian.

Theorem: (Dobrowolski, G.) Every left-ordered dp-minimal group is Abelian.

There are examples of non-Abelian, left-ordered, dp-rank-2 groups:

e.g. $G = \langle x, y : x^{-1}yx = y^{-1} \rangle$, ordered so that $x^n y^m \leq x^{n'} y^{m'}$ iff $(n, m) \leq (n', m')$ in the lexicographic order.

Abelianity and left-invariant orders

Theorem: (Simon) Every bi-ordered dp-minimal group is Abelian.

Theorem: (Dobrowolski, G.) Every left-ordered dp-minimal group is Abelian.

There are examples of non-Abelian, left-ordered, dp-rank-2 groups:

e.g. $G = \langle x, y : x^{-1}yx = y^{-1} \rangle$, ordered so that $x^n y^m \leq x^{n'} y^{m'}$ iff $(n, m) \leq (n', m')$ in the lexicographic order.

Pure ordered groups

Question: which ordered Abelian groups $\mathscr{G} = (G, <, +)$ in the language $\{<, +\}$ are dp-minimal?

Theorem: (Jahnke, Simon, Walsberg '15) $(G, <, +)$ is dp-minimal if and only if there are only finitely many primes p such that $[G : pG]$ is infinite.

Examples: $(\mathbb{Z}, <, +)$ and $(\mathbb{Q}, <, +)$ are dp-minimal, as well as

$$\mathbb{Z}_{(p)} = \{\frac{a}{b} \in \mathbb{Q} : b \text{ is relatively prime to } p\}$$

with the natural ordering as a subset of $\mathbb{Q}$.

Pure ordered groups

Question: which ordered Abelian groups $\mathscr{G} = (G, <, +)$ in the language $\{<, +\}$ are dp-minimal?

Theorem: (Jahnke, Simon, Walsberg '15) $(G, <, +)$ is dp-minimal if and only if there are only finitely many primes p such that $[G : pG]$ is infinite.

Examples: $(\mathbb{Z}, <, +)$ and $(\mathbb{Q}, <, +)$ are dp-minimal, as well as

$$\mathbb{Z}_{(p)} = \{\frac{a}{b} \in \mathbb{Q} : b \text{ is relatively prime to } p\}$$

with the natural ordering as a subset of $\mathbb{Q}$.

Pure ordered groups

Question: which ordered Abelian groups $\mathscr{G} = (G, <, +)$ in the language $\{<, +\}$ are dp-minimal?

Theorem: (Jahnke, Simon, Walsberg '15) $(G, <, +)$ is dp-minimal if and only if there are only finitely many primes p such that $[G : pG]$ is infinite.

Examples: $(\mathbb{Z}, <, +)$ and $(\mathbb{Q}, <, +)$ are dp-minimal, as well as

$$\mathbb{Z}_{(p)} = \{\frac{a}{b} \in \mathbb{Q} : b \text{ is relatively prime to } p\}$$

with the natural ordering as a subset of $\mathbb{Q}$.

Tame topology for DOAGs

Theorem (G.) If $\mathscr{G} = (G, <, +, \ldots)$ is a dp-minimal densely ordered Abelian group and $X \subseteq G$ is definable and nowhere dense, then X is finite.

A dp-minimal densely-ordered Abelian group may have dense-codense definable subsets: e.g. if $G = \mathbb{Z}_{(p)}$ with the usual ordering, pG is a dense subgroup.

Theorem (Simon): If $(G, <, +, \ldots)$ is a **divisible** ordered Abelian group, then any infinite definable $X \subseteq G$ has nonempty interior.

Corollary: An expansion of $(\mathbb{R}, <, +)$ is dp-minimal if and only if it is o-minimal.

Tame topology for DOAGs

Theorem (G.) If $\mathscr{G} = (G, <, +, \dots)$ is a dp-minimal densely ordered Abelian group and $X \subseteq G$ is definable and nowhere dense, then X is finite.

A dp-minimal densely-ordered Abelian group may have dense-codense definable subsets: e.g. if $G = \mathbb{Z}_{(p)}$ with the usual ordering, pG is a dense subgroup.

Theorem (Simon): If $(G, <, +, \dots)$ is a **divisible** ordered Abelian group, then any infinite definable $X \subseteq G$ has nonempty interior.

Corollary: An expansion of $(\mathbb{R}, <, +)$ is dp-minimal if and only if it is o-minimal.

Tame topology for DOAGs

Theorem (G.) If $\mathscr{G} = (G, <, +, \ldots)$ is a dp-minimal densely ordered Abelian group and $X \subseteq G$ is definable and nowhere dense, then X is finite.

A dp-minimal densely-ordered Abelian group may have dense-codense definable subsets: e.g. if $G = \mathbb{Z}_{(p)}$ with the usual ordering, pG is a dense subgroup.

Theorem (Simon): If $(G, <, +, \ldots)$ is a **divisible** ordered Abelian group, then any infinite definable $X \subseteq G$ has nonempty interior.

Corollary: An expansion of $(\mathbb{R}, <, +)$ is dp-minimal if and only if it is o-minimal.

Tame topology for DOAGs

Theorem (G.) If $\mathscr{G} = (G, <, +, \ldots)$ is a dp-minimal densely ordered Abelian group and $X \subseteq G$ is definable and nowhere dense, then X is finite.

A dp-minimal densely-ordered Abelian group may have dense-codense definable subsets: e.g. if $G = \mathbb{Z}_{(p)}$ with the usual ordering, pG is a dense subgroup.

Theorem (Simon): If $(G, <, +, \ldots)$ is a **divisible** ordered Abelian group, then any infinite definable $X \subseteq G$ has nonempty interior.

Corollary: An expansion of $(\mathbb{R}, <, +)$ is dp-minimal if and only if it is o-minimal.

Definable functions in DOAGs: piecewise continuity

Recall: If $(G, <, +, \ldots)$ is a densely ordered o-minimal group, then for any definable unary $f : G \to G$, there is a definable partition $G = I_1 \cup \ldots \cup I_n$ into intervals such that each $f \restriction I_i$ is continuous and strictly monotone or constant.

Theorem: (G.) If $(G, <, +, \ldots)$ is a densely ordered OAG and $f : G \to G$ is definable, then there is a definable partition $G = X_1 \cup \ldots \cup X_n$ such that every $f \restriction X_i$ is continuous.

But what about monotonicity?

Definable functions in DOAGs: piecewise continuity

Recall: If $(G, <, +, \ldots)$ is a densely ordered o-minimal group, then for any definable unary $f : G \to G$, there is a definable partition $G = I_1 \cup \ldots \cup I_n$ into intervals such that each $f \restriction I_i$ is continuous and strictly monotone or constant.

Theorem: (G.) If $(G, <, +, \ldots)$ is a densely ordered OAG and $f : G \to G$ is definable, then there is a definable partition $G = X_1 \cup \ldots \cup X_n$ such that every $f \restriction X_i$ is continuous.

But what about monotonicity?

Definable functions in DOAGs: piecewise continuity

Recall: If $(G, <, +, \ldots)$ is a densely ordered o-minimal group, then for any definable unary $f : G \to G$, there is a definable partition $G = I_1 \cup \ldots \cup I_n$ into intervals such that each $f \restriction I_i$ is continuous and strictly monotone or constant.

Theorem: (G.) If $(G, <, +, \ldots)$ is a densely ordered OAG and $f : G \to G$ is definable, then there is a definable partition $G = X_1 \cup \ldots \cup X_n$ such that every $f \restriction X_i$ is continuous.

But what about monotonicity?

Definable functions in DOAGs: local monotonicity

F.V. Kuhlmann: there are weakly o-minimal (hence dp-minimal) "centripetal contraction groups" with definable $\sigma : G \to G$ which is locally constant but not constant, so we cannot hope for full monotonicity.

Attempted generalization: If $f : G \to G$ is definable in a dp-minimal divisible OAG is for a partition $X_1 \cup \ldots \cup X_n$ such that each $f \restriction X_i$ is **locally monotonic**: for every $a \in X_i$ there is an $\epsilon > 0$ such that the restriction to $(a - \epsilon, a + \epsilon)$ is strictly increasing, strictly decreasing, or constant.

Definable functions in DOAGs: local monotonicity

F.V. Kuhlmann: there are weakly o-minimal (hence dp-minimal) "centripetal contraction groups" with definable $\sigma : G \to G$ which is locally constant but not constant, so we cannot hope for full monotonicity.

Attempted generalization: If $f : G \to G$ is definable in a dp-minimal divisible OAG is for a partition $X_1 \cup \ldots \cup X_n$ such that each $f \restriction X_i$ is **locally monotonic**: for every $a \in X_i$ there is an $\epsilon > 0$ such that the restriction to $(a - \epsilon, a + \epsilon)$ is strictly increasing, strictly decreasing, or constant.

Definable sets and functions in DOAGs: pathologies

Definable sets and functions in dp-minimal, densely ordered Abelian groups are nice locally, but can have global pathologies.

Example: (Dolich, G.) There is a dp-minimal, divisible ordered Abelian group $(G, <, +, P)$ such that P is a unary predicate for an open set with infinitely many convex components.

Furthermore, there seem to be divisible dp-minimal OAGs $(G, <, +, f)$ with $f : G \to G$ which is not monotonic on any open interval! (Work in progress with V. Verbovskiy.)

Theorem: (G., Verbovskiy) If $(G, <, +, \ldots)$ is a dp-minimal densely ordered Abelian group in which only boundedly many convex subgroups of G are definable, then any definable $f : G \to G$ can be decomposed into finitely many continuous, locally monotonic functions.

Definable sets and functions in DOAGs: pathologies

Definable sets and functions in dp-minimal, densely ordered Abelian groups are nice locally, but can have global pathologies.

Example: (Dolich, G.) There is a dp-minimal, divisible ordered Abelian group $(G, <, +, P)$ such that P is a unary predicate for an open set with infinitely many convex components.

Furthermore, there seem to be divisible dp-minimal OAGs $(G, <, +, f)$ with $f : G \to G$ which is not monotonic on any open interval! (Work in progress with V. Verbovskiy.)

Theorem: (G., Verbovskiy) If $(G, <, +, \ldots)$ is a dp-minimal densely ordered Abelian group in which only boundedly many convex subgroups of G are definable, then any definable $f : G \to G$ can be decomposed into finitely many continuous, locally monotonic functions.

Definable sets and functions in DOAGs: pathologies

Definable sets and functions in dp-minimal, densely ordered Abelian groups are nice locally, but can have global pathologies.

Example: (Dolich, G.) There is a dp-minimal, divisible ordered Abelian group $(G, <, +, P)$ such that P is a unary predicate for an open set with infinitely many convex components.

Furthermore, there seem to be divisible dp-minimal OAGs $(G, <, +, f)$ with $f : G \to G$ which is not monotonic on any open interval! (Work in progress with V. Verbovskiy.)

Theorem: (G., Verbovskiy) If $(G, <, +, \ldots)$ is a dp-minimal densely ordered Abelian group in which only boundedly many convex subgroups of G are definable, then any definable $f : G \to G$ can be decomposed into finitely many continuous, locally monotonic functions.

Definable sets and functions in DOAGs: pathologies

Definable sets and functions in dp-minimal, densely ordered Abelian groups are nice locally, but can have global pathologies.

Example: (Dolich, G.) There is a dp-minimal, divisible ordered Abelian group $(G, <, +, P)$ such that P is a unary predicate for an open set with infinitely many convex components.

Furthermore, there seem to be divisible dp-minimal OAGs $(G, <, +, f)$ with $f : G \to G$ which is not monotonic on any open interval! (Work in progress with V. Verbovskiy.)

Theorem: (G., Verbovskiy) If $(G, <, +, \ldots)$ is a dp-minimal densely ordered Abelian group in which only boundedly many convex subgroups of G are definable, then any definable $f : G \to G$ can be decomposed into finitely many continuous, locally monotonic functions.

Discrete OAGs: convex subgroups

Theorem (Walsberg '19) If $\mathscr{G} = (G, <, +, \ldots)$ is a dp-minimal discretely ordered group, then $\mathscr{G}$ is interdefinable with a model of Presburger arithmetic ($\mathrm{Th}(\mathbb{Z}, <, +)$) if and only if there are no nontrivial convex subgroups of G definable in $\mathscr{G}$.

Definable functions in OAGs: "continuity"

Let $(G, <, +, \ldots)$ be a $|T|^+$-saturated, discretely ordered dp-minimal Abelian group, and let O be the minimal nontrivial convex subgroup.

Note: $O \cong \mathbb{Z}$ and G/O is densely ordered.

Repeating arguments from the densely ordered case, we can probably show:

Conjecture: If $f : G \to G$ is definable and

$$\widetilde{f} = \{(a + O, f(a) + O) \, : \, a \in G\},$$

then $\widetilde{f}$ is the union of finitely many graphs of continuous functions.

Definable functions in OAGs: "continuity"

Let $(G, <, +, \ldots)$ be a $|T|^+$-saturated, discretely ordered dp-minimal Abelian group, and let O be the minimal nontrivial convex subgroup.

Note: $O \cong \mathbb{Z}$ and G/O is densely ordered.

Repeating arguments from the densely ordered case, we can probably show:

Conjecture: If $f : G \to G$ is definable and

$$\widetilde{f} = \{(a + O, f(a) + O) \,:\, a \in G\},$$

then $\widetilde{f}$ is the union of finitely many graphs of continuous functions.

Finite dp-rank OAGs

Question: Which ordered Abelian groups $(G, <, +)$ have finite dp-rank?

Theorem: For an ordered Abelian group $\mathscr{G} = (G, <, +)$, the following are equivalent:

1. $\mathscr{G}$ has finite dp-rank.
2. $\mathscr{G}$ is strong.
3. There are only finitely many primes p such that $[G : pG] = \infty$, and G has boundedly many definable convex subgroups.

(This was found independently by Hasson-Halevi, Dolich-G., and Farré.)

Example of G with unboundedly many definable convex subgroups: $\bigoplus_{i \in \omega} \mathbb{Z}_{(p)}$ with lexicographic ordering.

Finite dp-rank OAGs

Question: Which ordered Abelian groups $(G, <, +)$ have finite dp-rank?

Theorem: For an ordered Abelian group $\mathscr{G} = (G, <, +)$, the following are equivalent:

1. $\mathscr{G}$ has finite dp-rank.
2. $\mathscr{G}$ is strong.
3. There are only finitely many primes p such that $[G : pG] = \infty$, and G has boundedly many definable convex subgroups.

(This was found independently by Hasson-Halevi, Dolich-G., and Farré.)

Example of G with unboundedly many definable convex subgroups: $\bigoplus_{i \in \omega} \mathbb{Z}_{(p)}$ with lexicographic ordering.

Finite dp-rank OAGs

Question: Which ordered Abelian groups $(G, <, +)$ have finite dp-rank?

Theorem: For an ordered Abelian group $\mathscr{G} = (G, <, +)$, the following are equivalent:

1. $\mathscr{G}$ has finite dp-rank.
2. $\mathscr{G}$ is strong.
3. There are only finitely many primes p such that $[G : pG] = \infty$, and G has boundedly many definable convex subgroups.

(This was found independently by Hasson-Halevi, Dolich-G., and Farré.)

Example of G with unboundedly many definable convex subgroups: $\bigoplus_{i \in \omega} \mathbb{Z}_{(p)}$ with lexicographic ordering.

Finite-rank DOAGs: definable discrete sets

Theorem: (Dolich, G.) Suppose that $(G, <, +, \ldots)$ is an Archimedean ordered Abelian group of finite dp-rank (or even strong) and $X \subseteq G$ is definable and discrete. Then X is a finite union of arithmetic sets (i.e. sets of the form $\{a + bn : n \in \mathbb{N}\}$, possibly with $b = 0$).

More can be said in the non-Archimedean case: e.g. definable discrete subsets of G cannot become "more and more spread out."

Finite-rank DOAGs: definable discrete sets

Theorem: (Dolich, G.) Suppose that $(G, <, +, \ldots)$ is an Archimedean ordered Abelian group of finite dp-rank (or even strong) and $X \subseteq G$ is definable and discrete. Then X is a finite union of arithmetic sets (i.e. sets of the form $\{a + bn : n \in \mathbb{N}\}$, possibly with $b = 0$).

More can be said in the non-Archimedean case: e.g. definable discrete subsets of G cannot become "more and more spread out."

More questions

- Under what conditions can we prove reasonable theorems about monotonicity or cell decomposition for definable functions in a dp-minimal ordered Abelian group? ... or an ordered Abelian group of finite dp-rank?
- Can we "classify" in any reasonable sense the dp-minimal expansions of $(\mathbb{Z}, +)$? Must they all arise from a valuation, ordering, or cyclic ordering compatible with the group operation?
- Same question, for other strongly minimal Abelian groups (e.g. $(\mathbb{Z}_2^{\omega}, +)$).

More questions

- Under what conditions can we prove reasonable theorems about monotonicity or cell decomposition for definable functions in a dp-minimal ordered Abelian group? ... or an ordered Abelian group of finite dp-rank?
- Can we "classify" in any reasonable sense the dp-minimal expansions of $(\mathbb{Z}, +)$? Must they all arise from a valuation, ordering, or cyclic ordering compatible with the group operation?
- Same question, for other strongly minimal Abelian groups (e.g. $(\mathbb{Z}_2^{\omega}, +)$).

More questions

- Under what conditions can we prove reasonable theorems about monotonicity or cell decomposition for definable functions in a dp-minimal ordered Abelian group? ... or an ordered Abelian group of finite dp-rank?
- Can we "classify" in any reasonable sense the dp-minimal expansions of $(\mathbb{Z}, +)$? Must they all arise from a valuation, ordering, or cyclic ordering compatible with the group operation?
- Same question, for other strongly minimal Abelian groups (e.g. $(\mathbb{Z}_2^\omega, +)$).

Selected bibliography

- "Some remarks on inp-minimal and finite burden groups," J. Dobrowolski and J. Goodrick, *Archive for Mathematical Logic* vol. 58 (2019).
- "Strong theories of ordered Abelian groups," A. Dolich and J. Goodrick, *Fundamenta Mathematicae* vol. 236 (2017).
- "Strongly dependent abelian groups and Henselian fields," Y. Halevi and A. Hasson, arXiv:1706.03376 (2017).
- "Partial results on dp-finite fields," W. Johnson, arXiv:1903.11322 (2019).
- "Additivity of the dp-rank," Kaplan, Onshuus, and Usvyatsov, arXiv:1109.1601 (2011).
- "On dp-minimal ordered structures," P. Simon, *Journal of Symbolic Logic*,
- "A family of dp-minimal expansions of $(\mathbb{Z}; +)$," M. C. Tran and E. Walsberg, 2017.