Is $\aleph_1$-categoricity Absolute in $L_{\omega_1,\omega}$? Laskowski Fest

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Sacks Dicta

"... the central notions of model theory are absolute and absoluteness, unlike cardinality, is a logical concept. That is why model theory does not founder on that rock of undecidability, the generalized continuum hypothesis, and why the Łos conjecture is decidable."

Gerald Sacks, 1972

Our question

Does Sacks dicta extend from $L_{\omega,\omega}$ to $L_{\omega_1,\omega}$?

The main thread here is work Chris, Saharon and I have doing for most of this decade.

The Absoluteness Issue

Shoenfield Absoluteness Lemma

Theorem (Shoenfield)

If

1. $V \subset V'$ are models of ZF with the same ordinals and
2. ϕ is a lightface Π^1_2 predicate of a set of natural numbers

then for any $A \subset N$, $V \models \phi(A)$ iff $V' \models \phi(A)$.

Note that this trivially gives the same absoluteness results for Σ^1_2-predicates.

Complexity of first order concepts

For example a first order theory T is unstable just if there is a formula $\phi(\mathbf{x}, \mathbf{y})$ such for every n

$$T \models (\exists \mathbf{x}_1, \ldots \mathbf{x}_n \exists \mathbf{y}_1, \ldots \mathbf{y}_n) \bigwedge_{i<j} \phi(\mathbf{x}_i, \mathbf{y}_j) \wedge \bigwedge_{i \geq j} \neg\phi(\mathbf{x}_i, \mathbf{y}_j)$$

ω-stability, superstability and $\aleph_1$-categoricity are Π^1_1.

Easy remark

The class of first order -sentences (formulas) is arithmetic, in fact recursive.

The class of satisfiable first order sentences is Π^0_1.

Existence of a model M of T with $|M| = \aleph_1$ is absolute between models of ZF with the same ordinals.

Remark

Slightly more complicated remark:
Similar absoluteness results hold between ω-models of set theory for basic syntax and semantics of $L_{\omega_1,\omega}$.
In particular, ω-stablility

From $L_{\omega_1,\omega}$ to 'first order'

1. $\phi \in L_{\omega_1,\omega} \rightarrow (T, \Gamma)$
2. complete $\phi \in L_{\omega_1,\omega} \rightarrow (T, Atomic)$

The models of a complete sentence in $L_{\omega_1,\omega}$ can be represented as:
$\boldsymbol{K}$ is the class of atomic models (realize only principal types) of a first order theory (in an expanded language).

$(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ is the class of atomic models of a first order theory under elementary submodel.

ω-stability

Definitions

$p \in S_{at}(A)$ if $a \models p$ implies Aa is atomic.
$\boldsymbol{K}$ is ω-stable if for every countable model M, $S_{at}(M)$ is countable.

But, there may be $A \subseteq M$, $p \in S_{at}(A)$ that has no extension to $S_{at}(M)$.

Chris's framework

Is the property:

'the class of atomic models of a complete first order theory T is $\aleph_1$-categorical'

an absolute property of T?

First order absoluteness

Theorem (Morley-Baldwin-Lachlan)

A first order theory T in a countable language is $\aleph_1$ categorical iff

1. T has no 2-cardinal models and
2. T is ω-stable.

1) is arithmetic and 2) is Π^1_1.

Fact

A first order theory T in a countable language whose class of atomic models satisfies 1) and 2) is $\aleph_1$-categorical.

I emphasize Morley because it is his direction:
'$\aleph_1$-categorical implies ω-stable' that is problematic for $L_{\omega_1,\omega}$.

The $L_{\omega_1,\omega}$ case

One Completely General Result

WGCH(λ): $2^\lambda < 2^{\lambda^+}$
Let $\boldsymbol{K}$ be an abstract elementary class (AEC).

Theorem

[WGCH (λ)] Suppose $\lambda \geq \mathrm{LS}(\boldsymbol{K})$ and $\boldsymbol{K}$ is λ-categorical. If amalgamation fails in λ there are 2^{λ^+} models in $\boldsymbol{K}$ of cardinality $\kappa = \lambda^+$.

Uses $[\hat{\Theta}_{\lambda^+}(S)]$ for many S.

λ-categoricity plays a fundamental role.

Definitely not provable in ZFC for AEC (but maybe for $L_{\omega_1,\omega}$).

Getting ω-stability

Theorem: Keisler/Shelah

1. *(Keisler) ZFC* If some uncountable model in $\boldsymbol{K}$ realizes uncountably many types (in a countable fragment) over $\emptyset$ then $\boldsymbol{K}$ has $2^{\aleph_1}$ models in $\aleph_1$.
2. *(Shelah)* $2^{\aleph_0} < 2^{\aleph_1}$ If $\boldsymbol{K}$ has $< 2^{\aleph_1}$ models of cardinality $\aleph_1$, then $\boldsymbol{K}$ is ω-stable.

Two uses of WCH

1. WCH implies AP in $\aleph_0$. Thus, if $\boldsymbol{K}$ is not ω-stable there is a countable model M and an uncountable $N \in \boldsymbol{K}$ which realizes uncountably many types over M.

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2. By Keisler, $\mathrm{Th}_M(M)$ has $2^{\aleph_1}$ models. From WCH we conclude $\mathrm{Th}(M)$ has $2^{\aleph_1}$ models.

Is WCH needed?

for amalgamation

1. Yes for AEC, even for $L_{\omega_1,\omega}(Q)$.
   There is a sentence in $L_{\omega_1,\omega}(Q)$ that under MA is $\aleph_1$-categorical but is not ω-stable and fails amalgamation in $\aleph_0$

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counting models over a countable set ????

Another route to ω-stability

Morley's original proof using the Skolem hull gives:

Theorem

If a complete first order theory has arbitrarily large models and is $\aleph_1$-categorical then it is ω-stable.

More generally,

Theorem

An $\aleph_1$-categorical atomic class $\boldsymbol{K}$ that has arbitrarily large models and amalgamation in $\aleph_0$ is ω-stable.

Understanding *Locally* the models of $\aleph_1$-categorical $L_{\omega_1,\omega}$-sentences

The class of models

$\boldsymbol{K}_T$ is the class of atomic models of the countable first order theory T.

Definition

The atomic class $\boldsymbol{K}_T$ is extendible if there is a pair $M \preceq N$ of countable, atomic models, with $N \neq M$.

Equivalently, $\boldsymbol{K}_T$ is extendible if and only if there is an uncountable, atomic model of T.

We assume throughout that $\boldsymbol{K}_T$ is extendible. We work in the monster model of T, which is usually not atomic.

A new notion of closure

Definition

An atomic tuple $\mathbf{c}$ is in the pseudo-algebraic closure of the finite, atomic set B ($\mathbf{c} \in \mathrm{pcl}(B)$) if
for every atomic model M such that $B \subseteq M$, and $M\mathbf{c}$ is atomic, $\mathbf{c} \subseteq M$.

When this occurs, and $\mathbf{b}$ is any enumeration of B and $p(\mathbf{x}, \mathbf{y})$ is the complete type of $\mathbf{cb}$, we say that $p(\mathbf{x}, \mathbf{b})$ *is pseudo-algebraic.*

Example I

Our notion, pcl of *algebraic* differs from the classical first-order notion of algebraic as the following examples show:

Example

Suppose that an atomic model M consists of two sorts. The U-part is countable, but non-extendible (e.g., U infinite, and has a successor function S on it, in which every element has a unique predecessor). On the other sort, V is an infinite set with no structure (hence arbitrarily large atomic models). Then, an element $x_0 \in U$ is not algebraic over $\emptyset$ in the normal sense but is in $\mathrm{pcl}(\emptyset)$.

Example II

Example

Let $L = A, B, \pi, S$ and T say that A and B partition the universe with B infinite, $\pi : A \rightarrow B$ is a total surjective function and S is a successor function on A such that every π-fiber is the union of S-components. K_T is the class of $M \models T$ such that every π-fiber contains exactly one S-component. Now choose elements $a, b \in M$ for such an M such that $a \in A$ and $b \in B$ and $\pi(a) = b$. Clearly, a is not algebraic over b in the classical sense, but $a \in \mathrm{pcl}(b)$.

Definability of pseudo-algebraic closure

Strong ω-homogeneity of the monster model of T yields:

Fact

If $p(\mathbf{x}, \mathbf{y})$ is the complete type of $\mathbf{cb}$, then

$$\mathbf{c} \in \mathrm{pcl}(\mathbf{b}) \quad \text{if and only if} \quad \mathbf{c}' \in \mathrm{pcl}(\mathbf{b}')$$

for any $\mathbf{c}'\mathbf{b}'$ realizing $p(\mathbf{x}, \mathbf{y})$.
In particular, the truth of $c \in \mathrm{pcl}(\mathbf{b})$ does not depend on an ambient atomic model.

Further, since a model which atomic over the empty set is also atomic over any finite subset, moving M to N we have:

Fact

If $\mathbf{c} \notin \mathrm{pcl}(B)$, witnessed by M then for every countable, atomic $N \supset B$, there is a realization $\mathbf{c}'$ of $p(\mathbf{x}, B)$ such that $\mathbf{c}' \not\subseteq N$.

Pseudo-minimal sets

Definition

1. A *possibly incomplete* type q over $\mathbf{b}$ is *pseudominimal* if for any finite, $\mathbf{b}^* \supseteq \mathbf{b}$, $\boldsymbol{a} \models q$, and $\mathbf{c}$ such that $\mathbf{b}^*\mathbf{c}\boldsymbol{a}$ is atomic, if $\mathbf{c} \subset \mathrm{pcl}(\mathbf{b}^*\boldsymbol{a})$, and $\mathbf{c} \notin \mathrm{pcl}(\mathbf{b}^*)$, then $\boldsymbol{a} \in \mathrm{pcl}(\mathbf{b}^*\mathbf{c})$.
2. M is pseudominimal if $x = x$ is pseudominimal in M.

I.e, pcl satisfies exchange (and more); we have a geometry.

'Density'

Definition

$\boldsymbol{K}_T$ satisfies *'density' of pseudominimal types* if for every atomic $\mathbf{e}$ and atomic type $p(\mathbf{e}, \mathbf{x})$ there is a $\mathbf{b}$ with $\mathbf{eb}$ atomic and $q(\mathbf{e}, \mathbf{b}, \mathbf{x})$ extending p such that q is pseudominimal.

Failing 'density'

Lemma

$\boldsymbol{K}_T$ fails 'density' of pseudominimal types if, after naming a finite tuple $\mathbf{e}$, there is a complete 1-type $\tilde{p}(x)$ over $\mathbf{e}$ such that for any finite, atomic $\mathbf{b}$ containing $\mathbf{e}$ and complete $q(\mathbf{e}, \mathbf{b}, \mathbf{x})$ extending $\tilde{p}$ there are a finite atomic $\mathbf{b}^* \supset \mathbf{b}$, $\boldsymbol{a} \models q$, and $\mathbf{c}$ such that
$\mathbf{b}^*\mathbf{c}\boldsymbol{a}$ is atomic, $\mathbf{c} \subset \mathrm{pcl}(\mathbf{b}^*\boldsymbol{a})$, $\mathbf{c} \notin \mathrm{pcl}(\mathbf{b}^*)$, and $\boldsymbol{a} \notin \mathrm{pcl}(\mathbf{b}^*)$, but $\boldsymbol{a} \notin \mathrm{pcl}(\mathbf{b}^*\mathbf{c})$.

I.e. pcl fails exchange locally.

Method: ‘Consistency implies Truth’

[BL16]
Let ϕ be a τ-sentence in $L_{\omega_1,\omega}(Q)$ such that it is consistent that ϕ has a model.
Let A be the countable ω-model of set theory, containing ϕ, that thinks ϕ has an uncountable model.

Construct B, an uncountable model of set theory, which is an elementary extension of A
such that B is correct about uncountability. Then the model of ϕ in B is actually an uncountable model of ϕ.

Building many atomic models of T by ultralimits of models of set theory

The first technical tool is the iterated generic elementary embedding induced by the nonstationary ideal on ω_1, which we will denote by $NS\omega_1$.

The ultrafilter

Forcing with the Boolean algebra $(\mathcal{P}(\omega_1)^M$ over a ZFC model M gives rise to an M-normal ultrafilter U on ω_1^M (i.e., every regressive function on ω_1^M in M is constant on a set in U).

The Ultrapower

Given such M and U, we can form the generic ultrapower $Ult(M,U)$, which consists of all functions in M with domain ω_1^M,

where for any two such functions f, g, and any relation R in $\{=,\in\}$, fRg in $Ult(M,U)$ if and only if $\{\alpha < \omega_1^M \mid f(\alpha)Rg(\alpha)\} \in U$.

Nota Bene

If M is countable, $Ult(M,U)$ is countable.

By convention, we identify the well-founded part of the ultrapower $\mathrm{Ult}(M,U)$ with its Mostowski collapse.

Iterations

Definition

Let M be a model of a sufficient finite fragment of ZFC and let γ be an ordinal less than or equal to ω_1.
An iteration J_X of M of length γ consists of models

$$M_\alpha : \quad (\alpha \leq \gamma),$$

sets

$$G_\alpha : \quad (\alpha < \gamma),$$

and a commuting family of elementary embeddings

$$j_{\alpha\beta} : M_\alpha \to M_\beta : \quad (\alpha \leq \beta \leq \gamma)$$

such that for J_X we choose the ultrafilter U_α such that $P^{M_\alpha} \in U_\alpha$ iff $\alpha \in X$.

Main Theorem

Goal Theorem

If $\boldsymbol{K}_T$ fails 'density of pseudominimal types' then $\boldsymbol{K}_T$ has $2^{\aleph_1}$ models of cardinality $\aleph_1$.

We prove this in two steps

1. Force to construct a model (M, E) of set theory in which a model of T codes model theoretic and combinatorial information sufficient to guarantee the non-isomorphism of its image in the different ultralimits.
2. Apply Skolem ultralimits of the models of set theory from 1) to construct $2^{\aleph_1}$ atomic models of T with cardinality $\aleph_1$ in V.

Proof sketch:

Fix a countable transitive model M of ZFC and choose $S \subseteq \omega_1^M \setminus \{0\}$ such that

$$(M, \in) \models \text{`}S \text{ is stationary/costationary'}$$

Construct $(I, <, P, E) \in \mathbf{I}^*$ is an $\aleph_1$ dense linear order with a unary predicate P and an equivalence relation E such that I/E is dense and both P/E and $\neg P/E$ are dense.

Theorem

Suppose $\delta(x)$ is a complete, non-pseudo algebraic formula with no pseudo-minimal extension. There is a c.c.c. forcing $\mathbb{Q}_I$ such that in $M[G]$, there is a full, atomic $N_I \models T$ and

Ultralimits of $M[G]$ by iterations J_X and J_Y give rise to non-isomorphic atomic models N_X and N_Y of T if $X - Y$ is stationary.

Extension

Definition

$\boldsymbol{K}$ is pcl-small if $S_{at}(\mathrm{pcl}(\boldsymbol{a}))$ is countable for every finite sequence $\boldsymbol{a}$.

[LS19] show:

Theorem

If $\boldsymbol{K}$ has fewer than $2^{\aleph_1}$ models in $\aleph_1$, then $\boldsymbol{K}$ is pcl-small.

Do $\aleph_1$-categorical theories have 'big' models

Must $\aleph_1$-categorical theories have a bounded number of models?

How big is big?

In the mid-70's Shelah answered my question as to whether a sentence of $L_{\omega_1,\omega}(Q)$ could be *categorical in the philosophers sense*, have only one model. In different papers he proved in different ways such a theory has a model in $\aleph_2$.

A natural question is whether a sentence of $L_{\omega_1,\omega}$ that is $\aleph_1$-categorical has a model in $2^{\aleph_0}$.

Getting models in $2^{\aleph_0}$: Method

In the novel *White Light* [Ruc80], Rudy Rucker proposes a metaphor for the continuum hypothesis. One can reach $\aleph_1$ by a laborious climb up the side of Mt. ON, pausing at ϵ_0.

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Cantor's elevator An instantaneous trip up a shaft at the center of the mountain.

For atomic models we take the slightly slower
Shelah's elevator The elevator is a bit slower but has only countably many floors. After building each finitely many rooms at each step we reach an object of cardinality $2^{\aleph_0}$.

Asymptotic similarity

Definition

Fix an L-structure M. A subset of M, indexed by $\{a_\eta : \eta \in 2^\omega\}$, is *asymptotically similar* if, for every k-ary L-formula θ, there is an integer N_θ such that for every $\ell \geq N_\theta$,

$$M \models \theta(a_{\eta_0}, \ldots, a_{\eta_{k-1}}) \leftrightarrow \theta(a_{\tau_0}, \ldots, a_{\tau_{k-1}})$$

whenever $(\eta_0, \ldots, \eta_{k-1})$ and $(\tau_0, \ldots, \tau_{k-1})$ are similar $(\mathrm{mod}\ \ell)$.

Remark

Asymptotic similarity is a type of indiscernibility, but, the indiscernibility is only formula by formula. Consider $M = (2^\omega, U_a)_{a \in 2^{<\omega}}$, where each U_a is a unary predicate interpreted as the cone above a, i.e., $U_a(M) = \{\eta \in 2^\omega : a \triangleleft \eta\}$. In M, the entire universe $\{\eta : \eta \in 2^\omega\}$ is asymptotically similar, despite the fact that no two elements have the same 1-type.

Getting models in $2^{\aleph_0}$

Theorem [BL19]

If a countable first order theory T has an atomic pseudominimal model M of cardinality $\aleph_1$ then there is an atomic pseudominimal model N of T which a contains a set of *asymptotically similar* elements with cardinality $2^{\aleph_0}$.
Equivalently, if the models of a complete sentence Φ in $L_{\omega_1,\omega}$ are pseudominimal and Φ has an uncountable model, it has a model in the continuum.

A simple application of the method gives Borel models in the continuum of any theory with trivial definable closure.

State of the art: Eventual behavior

[SV18] (simplified for ease of reading)

Theorem

Let κ be a strongly compact cardinal and let ψ be an $L_{\kappa,\omega}$-sentence. If ψ is categorical in some $\mu \geq \beth_{(2^\kappa)^+}$, then ψ is categorical in all $\mu \geq \beth_{(2^\kappa)^+}$

Extending [She83a, She83b]:

For any countable AEC, very few models below $\aleph_\omega$ and $2^{\aleph_n} < 2^{\aleph_{n+1}}$ for $n < \omega$ imply

1. excellence, thus arbitrarily large models,
2. and so categoricity in any cardinal implies categoricity in all uncountable cardinals.

very few: $I(\boldsymbol{K}, \aleph_n) \leq 2^{\aleph_{n-1}}$ for $n < \omega$.

What is the right analogy?

1. The weak diamond WGCH theory
2. Zilber quasiminimality

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3. uncountable first order theories: 'superstable' but not necessarily ω-stable.
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5. The MA '$\aleph_1$-categoricity doesn't yield structure' approach.

Questions

Let ϕ be a complete sentence of $L_{\omega_1,\omega}$.

1. If ϕ is $\aleph_1$-categorial, must ϕ have arbitrarily large models?
2. If ϕ is $\aleph_1$-categorial, must ϕ be ω-stable?
3. If ϕ characterizes $\kappa > \aleph_0$ must ϕ have 2^κ models in κ?
4. For $\kappa < \aleph_{\omega_1}$, describe an explicit sentence that characterizes κ. [BKL17]

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